Pseudo-Transient Continuation for Variably Saturated Flow in Porous Media

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Solution Methods for Large Scale Nonlinear Problems

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Outline

- Model Equations For Variably Saturated Flow (Richards Equation).
- Iterative Methods for Obtaining the Steady-State Solutions.
- Test Problems and Computational Results.

Richards Equation

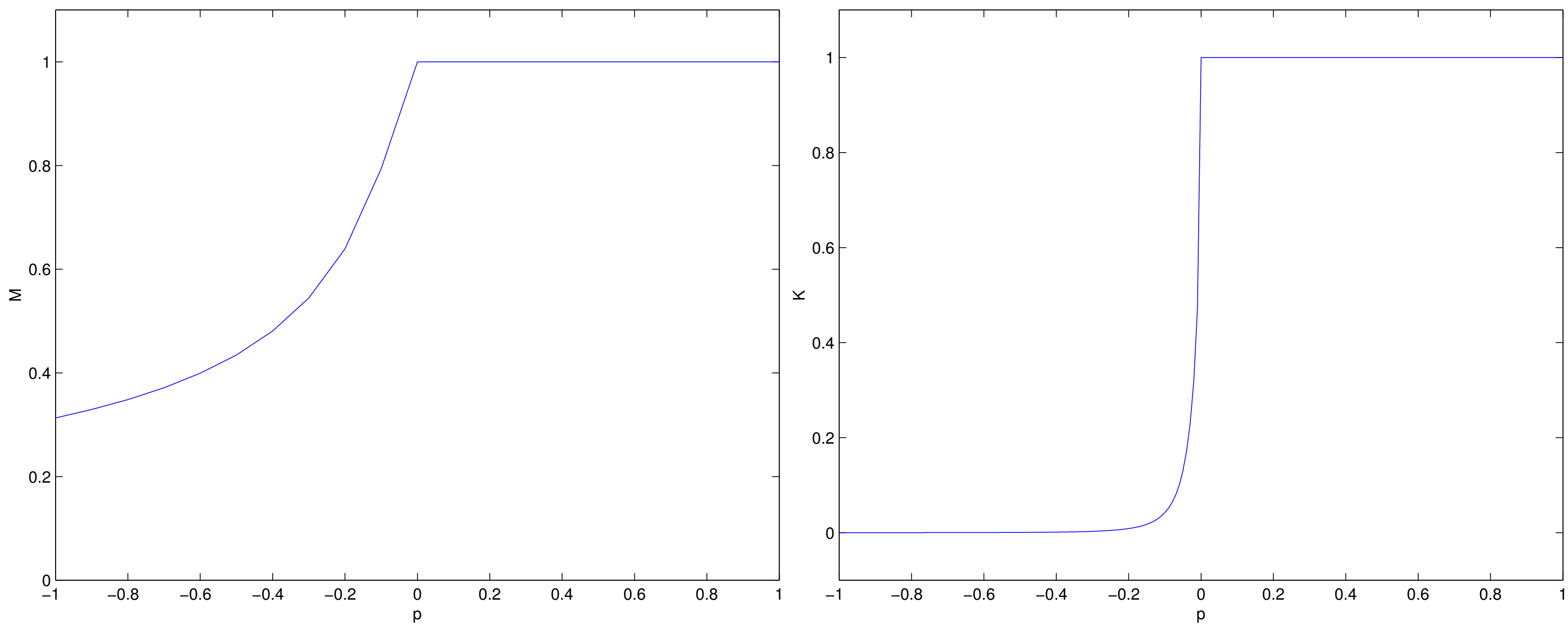
Find $p : \Omega \times I \rightarrow \mathbb{R}$ such that

$$\frac{\partial[m(p)]}{\partial t} = -\nabla \cdot [\mathsf{K}(p)\nabla p + \mathbf{g}(p)]$$

subject to boundary and initial conditions

$$\begin{aligned} p &= p_d && (\mathbf{x}, t) \in \partial\Omega_d \times I \\ (\mathsf{K}(p)\nabla p + \mathbf{g}(p)) \cdot \eta &= u_n && (\mathbf{x}, t) \in \partial\Omega_n \times I \\ p(\mathbf{x}, 0) &= p_0 && \mathbf{x} \in \Omega \end{aligned}$$

Nonlinearities



- $\frac{\partial m}{\partial p} = 0, \infty$ and is discontinuous.
- $\frac{\partial \mathbf{K}}{\partial p} = 0, \infty$ and is discontinuous.
- $\lim_{p \to -\infty} \mathbf{K}(p) = 0$.

Fully Discrete Equations

- Spatial Discretization + Backward Euler yields numerical solution at $t^{n+1} = t^n + \Delta t$:

$$\hat{F}(P^{n+1}) = \frac{1}{\Delta t} M(P^{n+1}) - \frac{1}{\Delta t} M^n - D(P^{n+1})P^{n+1} - G(P^{n+1}) = 0$$

- Or for the steady state we have

$$F(P) = -D(P)P - G(P) = 0$$

Steady-State Iterations

- $P_c =$ current iterate; $P_+ =$ next iterate.
- Fixed Point Iteration (Picard Iteration):

$$\begin{aligned} P_+ &= -D^{-1}(P_c)G(P_c) \\ &= P_c - [-D^{-1}(P_c)]F(P_c) \end{aligned}$$

- Newton's Method:

$$\begin{aligned} P_+ &= P_c - F'^{-1}(P_c)F(P_c) \\ &= P_c - [-D(P_c) - D'(P_c) - G'(P_c)]^{-1}F(P_c) \end{aligned}$$

Pseudo-Transient Continuation

- Start with Newton's method for Backward Euler

$$P_{+}^{n+1} = P_{c}^{n+1} - \hat{F}'^{-1}(P_{c}^{n+1})\hat{F}(P_{c}^{n+1})$$

- If $P_{c}^{n+1} = P^{n}$ the first iterate is

$$P_{+} = P_{c} - [\frac{1}{\Delta t}M'(P_{c}) + F'(P_{c})]^{-1}F(P_{c})$$

- As $\Delta t \rightarrow \infty$ this iteration approaches the steady-state Newton iteration.
- Otherwise it is a conditionally stable time integration method for the time-dependent problem (Rosenbrock 1963).

Pseudo-Transient Continuation,cont'd

- Pseudo-Transient Continuation consists of the Rosenbrock method + an adaptive selection of Δt such that $\Delta t \to \infty$ as $F \to 0$.
  1. Switched Evolution Relaxation (SER):

$$\Delta t_+ = \frac{\|F_+\|_2}{\|F_c\|_2} \Delta t_c$$

  2. Temporal Truncation Error (TTE): Choose Δt_+ such that

$$\left|\frac{(\Delta t_+)^2}{2(1+|M_{c,i}|)} \frac{\partial^2 M_{c,i}}{\partial t^2}\right| < \tau \quad i = 0, \ldots, N$$

     where i is the nodal index and $\frac{\partial^2 M_{c,i}}{\partial t^2}$ is approximated.

Steady State Iterations: Review

- Picard

$$P_+ = P_c - [-D^{-1}(P_c)]F(P_c)$$

- Newton

$$P_+ = P_c - [-D(P_c) - D'(P_c) - G'(P_c)]^{-1}F(P_c)$$

- Ψtc

$$P_+ = P_c - [\frac{1}{\Delta t}M'(P_c) - D(P_c) - D'(P_c) - G'(P_c)]^{-1}F(P_c)$$

Variations/Globalization

- Newton-Picard

$$P_+ = P_c - [-D(P_c) - \lambda_{np}(D'(P_c) + G'(P_c))]^{-1} F(P_c)$$

$\lambda_{np} \to 1$ as $F \to 0$

- Damped Newton (line search)

$$P_+ = P_c - \lambda_{ls}[-D(P_c) - D'(P_c) - G'(P_c)]^{-1} F(P_c)$$

$\lambda_{ls} \to 1$ as $F \to 0$

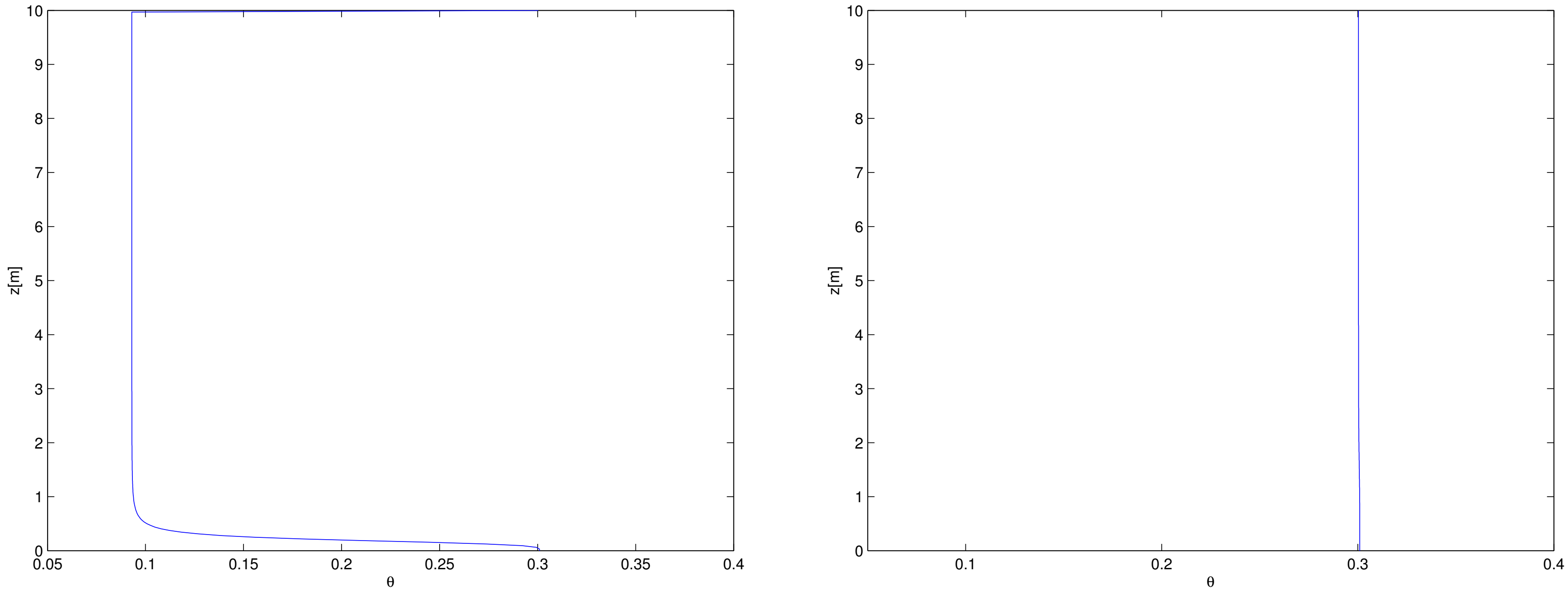
Theoretical Game Plan

- Use "Global" facts to ensure that we get into the region of Newton convergence.
- Picard-Newton: $D^{-1}F$ is a contraction therefore $\{P_n\}$ converges to the unique fixed point.
- Damped-Newton: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence, and once we're in the Newton ball we have to stay there.
- Ψtc : The time-dependent problem has a stable steady state, and the integration method is stable for the $\{\Delta t_n\}$.

Test Problems

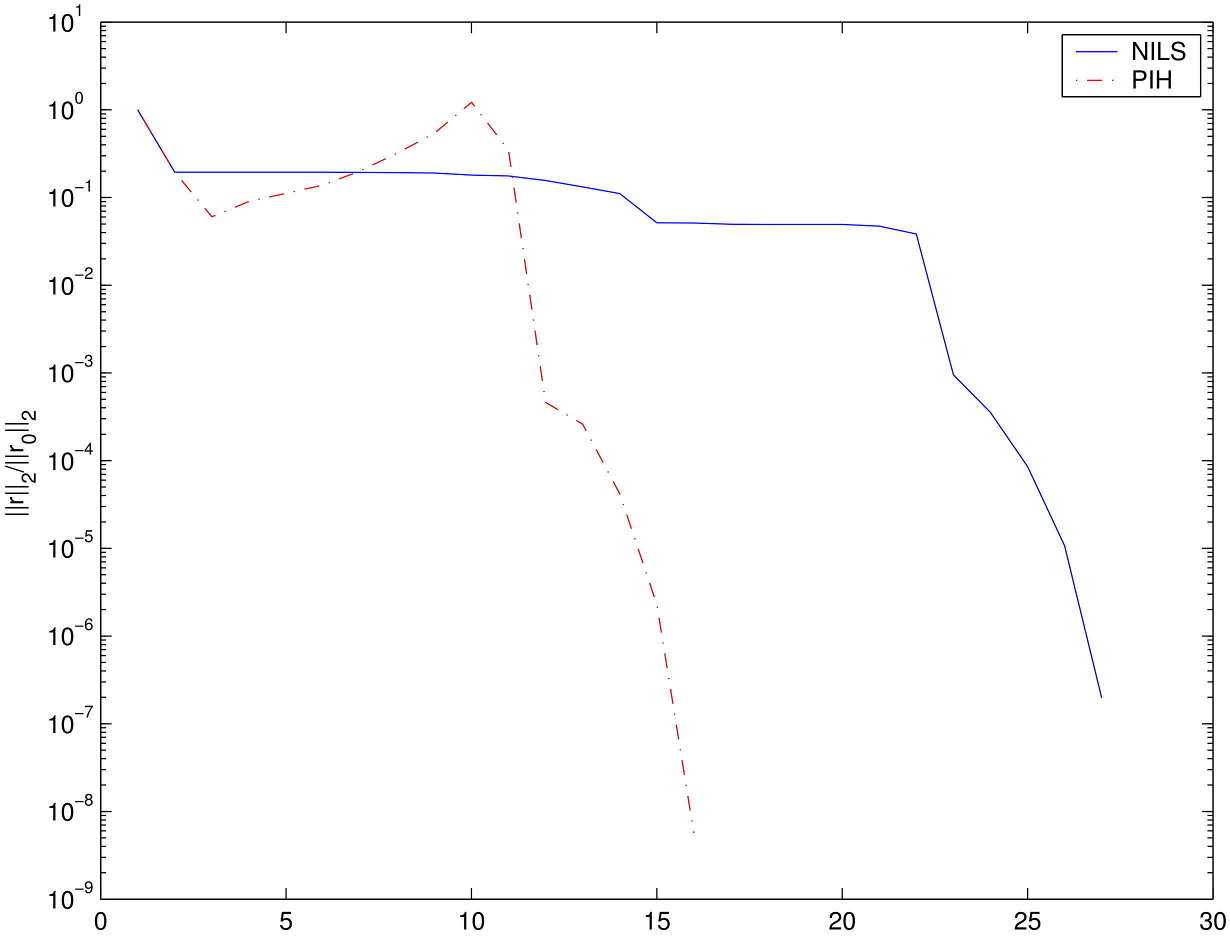
- 1D and 2D infiltration problems.
- The reference solution was an accurate time integration to steady-state using BDF methods.
- We compared Picard-Newton,Damped Newton, and Ψtc (TTE and SER).
- The switching criterion for Picard-Newton was $\|F_c\| \leq .01\|F_0\|$.
- The damped Newton used an Armijo linesearch based on a quadratic model.
- We tested LU, ILU-BiCGstab, and Two-Level Hybrid Additive Schwarz-BiCGstab.

1D Test Problem



Initial Conditions (left); Solution (right)

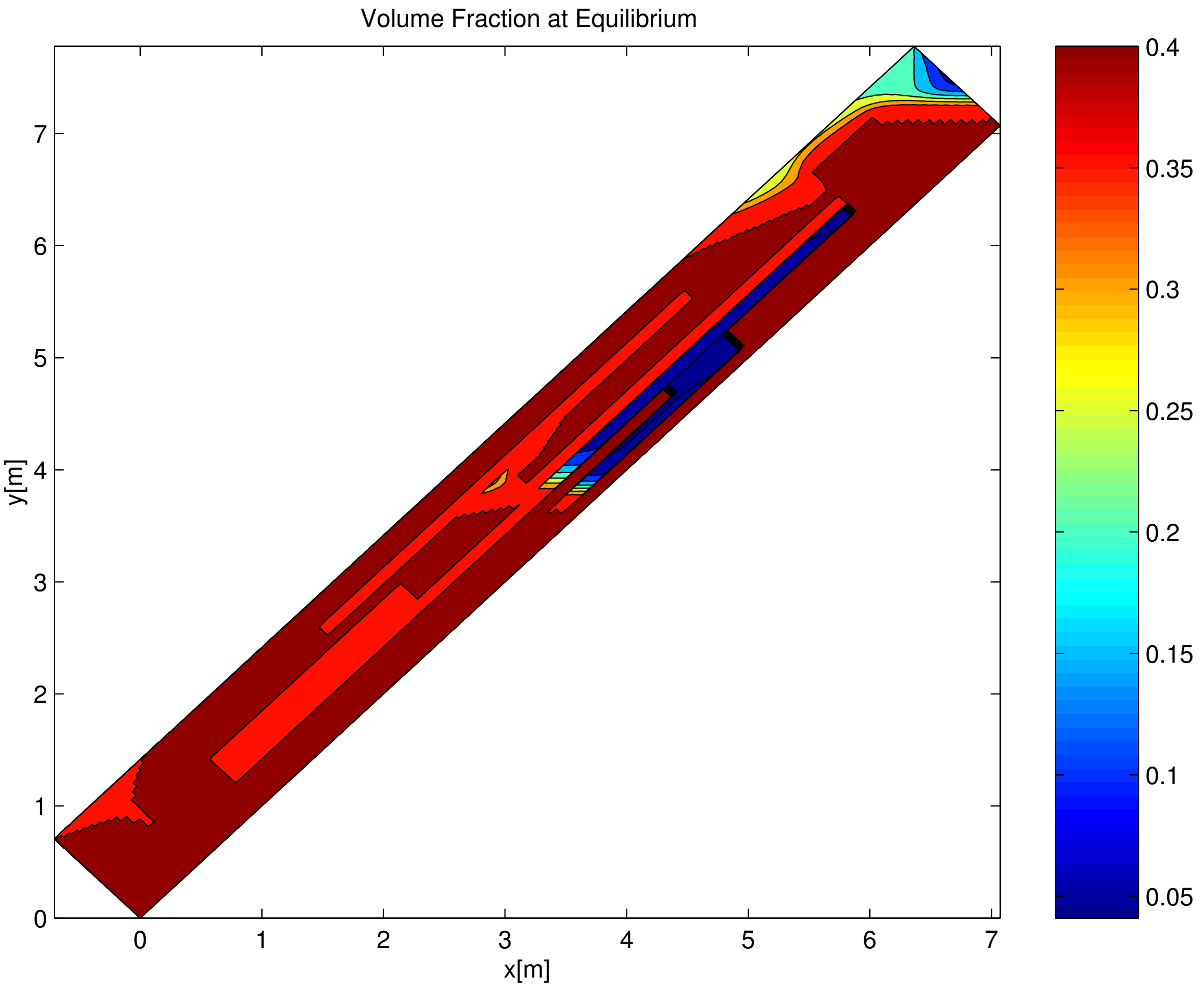
Residual History



Results

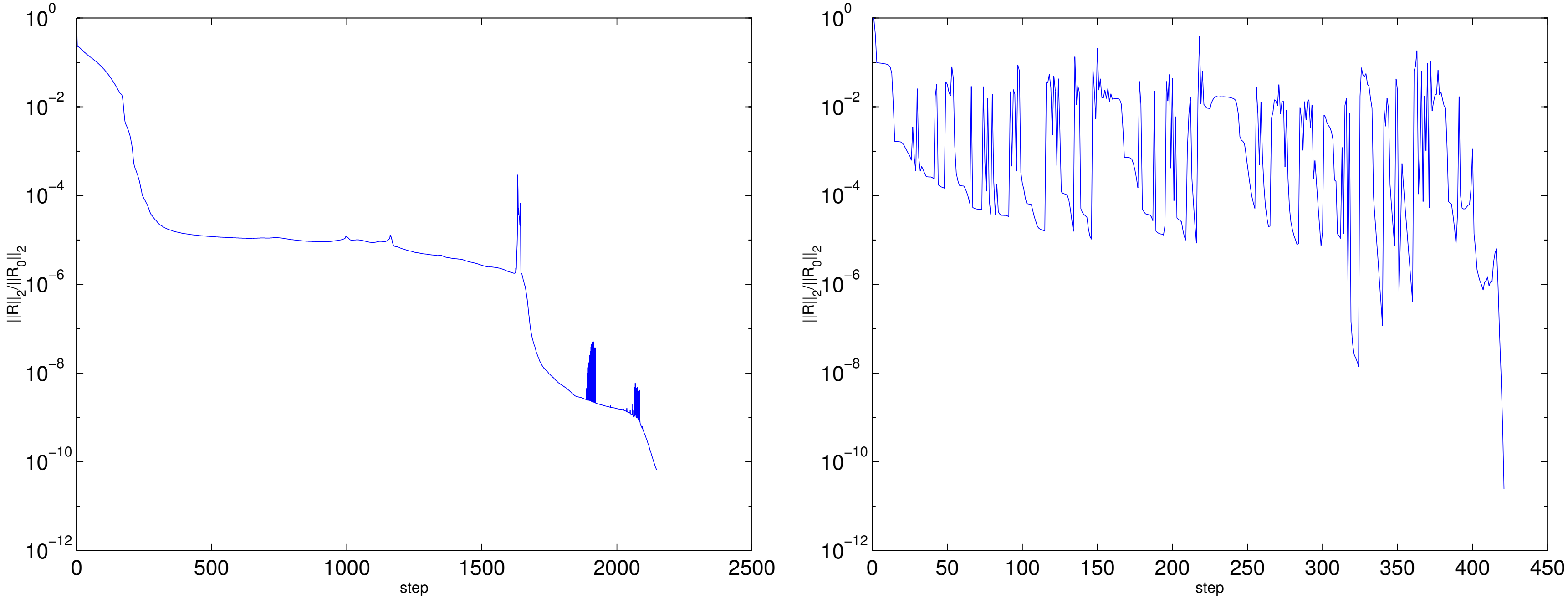
	Jeval	Feval	Steps	NLI	NLF	LI	LF	LS	Clock
				$n_e = 41$					
NILS	30	299	30	30	0	0	0	102	0.05
PIH	14	33	14	14	0	0	0	1	0.01
TTE	51	103	51	0	0	0	0	0	0.06
SER	83	167	83	0	0	0	0	0	0.04
				$n_e = 161$					
NILS	25	329	25	25	0	0	0	128	0.08
PIH	16	35	16	16	0	0	0	0	0.06
TTE	207	410	202	0	0	0	0	0	1.43
SER	353	705	351	0	0	0	0	0	0.33

2D Test Problem



Solution

Residual History



SER (left); TTE (right)

Results

	n_e	Jeval	Feval	Steps	LI	LF	LS	Clock
LU-PP								
TTE	11×11	114	229	114	114	0	0	1.57
TTE	21×21	127	255	127	127	0	0	2.96
TTE	41×41	125	250	124	125	0	0	8.16
TTE	81×81	146	281	134	146	0	0	58
TTE	161×161	248	458	209	248	0	0	667.07
NILS	161×161	232	12961	233	232	0	6043	429.58
BiCGstab-HAS								
TTE	11×11	115	231	115	979	0	0	1.86
TTE	21×21	125	251	125	2173	0	0	4.28
TTE	41×41	131	262	130	3782	0	0	18.17
TTE	81×81	152	288	135	7192	0	0	131.9
TTE	161×161	283	472	188	20806	0	0	1386.07
NILS	161×161	236	13153	237	70484	0	6130	3397

Conclusions

- When they work, Damped Newton and Newton-Picard are very fast.
- Ill-conditioning of D (or F') can be a source of trouble.
- Ψtc is more robust with respect to ill-conditioning–apparently because of the addition of a $1/\Delta t$ term to the diagonal.
- In practice Ψtc with TTE is violating the assumptions of the theory behind Ψtc (Δt is unstable), but it works anyway as long as we bracket.