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George Chapline

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MAP COLORING AND CALABI–YAU VACUA*

George Chapline

Physics Department
Lawrence Livermore National Laboratory
University of California
Livermore, CA 94550

ABSTRACT

An algorithm for coloring planar graphs is exploited to reveal a connection between Calabi–Yau manifolds and solutions of map coloring problems on a Riemann surface. This connection may shed light on the connection between superstring vacua and topological quantum field theories.

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INTRODUCTION

It has been suggested [1] that the most realistic superstring theories are those whose vacuum state contains a Calabi–Yau manifold. Unfortunately, there are thousands of candidate Calabi–Yau manifolds (as well as many other possibilities for the vacuum state such as orbifolds). This lead to the question of what ingredients are necessary to pick out some very special Calabi–Yau manifold as the particular vacuum state that occurs in Nature. It has recently been shown [2] that Calabi–Yau manifolds can arise as the fixed points of renormalization group flows for Landau–Ginzburg models. In this letter we point out that Calabi–Yau manifolds also arise naturally when one considers the problem of map coloring on a Riemann surface. In particular, we will show that in the continuum limit certain map coloring algorithms will constrain complex colors to lie on a Calabi–Yau manifold. As was the case for the Landau–Ginzburg models the appearance of a Calabi–Yau manifold is a consequence of the introduction of projective coordinates. In the case of map coloring this has the intuitively appealing interpretation that one is only interested in the coloring pattern, and not the absolute value of the colors. From this point of view the vanishing of the first Chern class is due to stability of the solution under "simulated annealing."

At first sight it may seem to be just a coincidence that classical stochastic equations of motion could yield the same result as renormalization group analysis of a quantum field theory. However, for 2–dimensional supersymmetric quantum field theories the Nicolai map [3] provides an equivalence between the 2–dimensional quantum field theory and a classical stochastic equation [4]. According to this equivalence functional integration over a scalar field ϕ and fermionic field ψ can be replaced by integration over a stochastic variable b, which can be interpreted as the noise source in a Langevin equation:

$$\frac{\partial\phi}{\partial t} + \frac{\delta S}{\delta\phi} = b, \tag{1}$$

where $S(\phi)$ is a classical action. The variable t in this equation is extraneous to the coordinates x_o, x_1 that appear in the original 2-dimensional field theory, and will be referred to in the following as stochastic time. In the problem of interest to us the dynamical system corresponding to Eq. (1) can be thought of as a "neural network" for coloring maps on a Riemann surface.

One advantage of replacing a 2-dimensional Landau-Ginzburg type quantum field theory with a neural network for coloring maps on a Riemann surface is that the importance of topological type solutions becomes immediately evident. It has been suggested [5,6] that superstring theories might emerge as the low temperature limit of an underlying theory whose high temperature behavior is purely topological. In the context of map coloring problems it is known [7] that there is an ultrametricity transition in the configuration space of solutions. For a Riemann surface with genus g there is a critical number N_c, known as the Heawood number, such that if the number of colors N is less than N_c then the solutions arrange themselves in a ultrametric way. For these ultrametric solutions the topology of the Riemann surface is important, and transitions between different solutions have the character of topological excitations in a 2 + 1 dimensional space, where the extra dimension corresponds to stochastic time.

E. Witten has recently pointed out [8] that there is an intimate connection between 2-dimensional conformal field theories and a topological quantum field theory in 2 + 1 dimensions that uses the Chern-Simons 3-form as the action. This connection closely parallels the connection between knot theory in 3-dimensions and 2-dimensional statistical models discovered by V. Jones [9]. In fact, the knot invariant discovered by Jones, the Jones polynomial, is related to the partition

function for an N–state Potts model [10]. It is well known that the antiferromagnetic Potts model reduces at zero temperature to the problem of coloring maps on a surface using N colors. Thus it is possible that the results described in this paper may shed some light on the relation between superstring vacua and topological quantum field theories in 2 + 1 dimensions.

In the next section we describe an algorithm for coloring maps on a Riemann surface that can be impleted by solving an equation of motion of the form (1). The b term in this context represents simulated annealing, and the condition that this term will have no effect is equivalent to the Calabi–Yau condition.

MAP COLORING ON A RIEMANN SURFACE

The map coloring algorithm that we will make use of was invented [11] to show how a neural network [12] can be constructed that will provide good solutions for map coloring problems. Since the solution to a map coloring problem on a Riemann surface is a pattern the solution should be invariant under conformal transformations of the Riemann surface. Thus the solution found by the neural network might be interpretable as the vacuum state of a string theory. In fact, as we will presently show in some interesting cases the neural network constrains the colors to lie on a Calabi–Yau manifold.

Let us consider a graph on the surface of a Riemann surface of genus g which has F regions, E edges, and V vertices. These quantities are related by Euler's classic relation

$$V - E + F = 2 - 2g \tag{2}$$

To every region we assign a color vector η_i^a, $i = 1, .., F$ and $a = 1, .., N$. Each color vector has N components, where N is the total number of colors we are going to use for the map coloring. For a surface with genus g the number of colors N would in general have to exceed the Heawood number, $N_c = [\frac{7}{2} + \frac{1}{2}(48g + 1)^{1/2}]$ [13], in order that 2 neighboring regions don't have the same color. However, when the number of regions F greatly exceeds the genus g, a fixed number of colors $N \leq 7$ will in most cases suffice. To see this we note that since $2E/V \geq 3$, Eq. (1) implies that

$$\frac{2E}{F} \leq 6(1 + \frac{2g - 2}{F}) \tag{3}$$

If $F >> g \geq 1$, then $2E/F \leq 6$. However, this implies that 7 colors will typically suffice, as can be seen by induction on F and g. Suppose that 7 colors suffice for maps with F regions on surfaces with genus $g' \leq g$ and suppose $F >> g$. Then Eq. (3) implies that if we add one more region we can still find a region with 6 or fewer adjacent regions. By hypothesis a satisfactory coloring exists for the map with this region deleted. Using the color that wasn't used to color the regions adjacent to this region it follows that a satisfactory coloring exists for the map with F + 1 regions. It should be noted that this argument holds whether or not the extra region contained a "handle" which increased the genus, because the presence or absence of a handle in a single region doesn't affect the coloring. In this way we can obtain a map with F + 1 regions and genus g + 1. We can extend this result to maps where the extra region is colored in an arbitrary way because 7 colors suffices to color a torus [13]. Of course, as shown in the celebrated work of Ringel and Young [14] there are some maps for every genus where the number of colors required is equal to N_c.

Suppose that we have N colors available to color maps on a surface of genus g. We may ask for an algorithm that determines a valid coloring on the surface.

Following Hopfield et al. [12] this may be accomplished [11] by minimizing a certain positive definite function $E(\eta)$ of the color vectors η_i. As shown in Ref. 11 this function may be taken to be a sum of 3 quartic polynomials.

$$E(\eta) = \alpha \sum_{i=1}^{F} (\vec{\eta}_i \bullet \vec{\eta}_i - 1)^2 + \beta \sum_{<ij>} (\vec{\eta}_i \bullet \vec{\eta}_j)^2$$

$$+ \gamma \sum_{i=1}^{F} [\vec{\eta}_i \bullet \vec{\eta}_i)^2 - \sum_{a=1}^{N} (\eta_i^a)^4] \quad , \tag{4}$$

where <ij> means a sum over nearest neighbor pairs. The first term on the r.h.s. of Eq. (4) is simply a normalization condition. Since from the point of view of finding a coloring pattern the absolute normalization of the color vectors is arbitrary it will be convenient in the following to introduce projective coordinates $\tilde{\eta}_i^a$, $a = 1, \cdots, N + 1$. It will also be convenient to complexity the color vectors. Expression (4) then becomes:

$$E(\tilde{\eta}) = \alpha \sum_{i=1}^{F} \sum_{a=1}^{N} (\tilde{\eta}_i^a \ \tilde{\eta}_i^a + \tilde{\eta}_i^{N+1} \ \tilde{\eta}_i^{N+1})^2$$

$$+ \beta \sum_{<ij>} \sum_{a=1}^{N} (\tilde{\eta}_i^a \ \tilde{\eta}_j^a)^2$$

$$+ \gamma \sum_{i=1}^{F} [(\sum_{a=1}^{N} \tilde{\eta}_i^a \ \tilde{\eta}_i^a)^2 - \sum_{a=1}^{N} (\tilde{\eta}_i^a \ \eta_i^a)^2] \tag{5}$$

The function $E(\tilde{\eta})$ will be minimized when each of the 3 terms on the r.h.s. of Eq. (5) vanish. This amounts to 2 conditions for each of the F regions and 1 condition for each of the E edges. If we denote the number of constraints per region by k then it follows from Eq. (3) that

$$k = 2 + \frac{E}{F} \leq 5 \tag{6}$$

when F >> g. The important point for us is that as a function of stochastic time the solutions of Eq. (1) will approach as closely as possible the zeros of $E(\tilde{\eta})$.

THE CALABI-YAU CONDITION

The process of minimizing the energy functional $E(\eta)$ will evidently constrain the color vectors η^a to lie on an algebraic submanifold of CP^N. We are interested here in the question whether this submanifold can be Ricci-flat; i.e., whether it is a Calabi-Yau manifold. The condition that an algebraic submanifold of CP^N be Ricci-flat is [1]:

$$\sum_{\alpha=1}^{n} d_\alpha = N + 1 \tag{7}$$

where the submanifold is specified by n homogeneous polynomials of degrees $d_1, \ldots, d_n$. If all the polynomials are quartic then condition (7) becomes

$$4n = N + 1, \tag{8}$$

We could have solutions to Eq's 6 and 8 for N = 7, N = 11 and 15. The corresponding solution spaces would have real codimensions 10, 16 and 22. On the other hand for a valid coloring solution the second term in Eq. (5) vanishes. Combining the first and third terms of Eq. (5) so that n = 1 then gives

$$N = 3 \tag{9}$$

As discussed in the last section in general N_c colors are required to color maps on a Riemann surface. Thus it might appear that the Calabi-Yau condition, Eq. (8),

is inconsistent with valid map colorings. On the other hand, there clearly are maps that can be colored with fewer than N_c, even 3 colors. For example, with certain exceptions (cf. [13]) triangulations of the Riemann surface can be 3-colored. We would conjecture that these 3-color graphs are required to obtain a sensible local field theoretical limit. Of course, attempts to color general maps with 3-colors will be frustrated, and the corresponding solutions to Eq. (1) will presumably be arranged ultrametrically as described in Ref. 7. For these maps only topological quantities have any significance.

The Nicolai map, Eq. (1), allows one to exchange functional integration over stochastic variables b_i for functional integration over <u>quantum</u> variables ϕ_i. The Jacobian that describes this change of variables can be written as a fermionic determinant:

$$\det\,(\delta b/\delta\psi) = \int [D\bar{\psi}]\,[D\psi]\; e^{\int dt d^2x\, \bar{\psi}\left(\frac{\partial}{\partial t} + \frac{\delta^2 S}{\delta\phi^2}\right)\psi} \tag{10}$$

where the fermion fields ψ_i are the supersymmetry partners of the ϕ_i. If we introduce chiral superfields Φ_i then the functional integrals over ϕ_i and ψ_i can be written in the form

$$\int [D\Phi_1] \cdots [D\Phi_{N+1}]\; e^{i \int d^2x d^2\theta [W(\Phi) + K(\Phi, \bar{\Phi})]} \tag{11}$$

where $K(\Phi, \bar{\Phi})$ are D-terms and $W(\Phi)$ is a superpotential. The renormalization group limit of expression (11) depends only on the form of the superpotential $W(\Phi)$; i.e. the kinetic energy terms are irrelevant operators.

In the context of map coloring the discussion in the previous section suggests that an appropriate choice for $W(\Phi)$ would be the first and third terms of Eq. (5); i.e.

$$W(\Phi) = (\Phi_1^2 + \cdots + \Phi_{N+1}^2)^2$$
$$+ (\Phi_1^2 + \cdots + \Phi_N^2)^2 - (\Phi_1^4 + \cdots + \Phi_N^4) \tag{12}$$

If we introduce variables

$$\zeta_{N+1} = \Phi_{N+1}^4, \quad \zeta_i = \left(\frac{\Phi_i}{\Phi_{N+1}}\right)^2, \quad i = 1, \cdots, N \tag{13}$$

the $W(\Phi)$ can be written in the form

$$W(\Phi) = \xi_{N+1}[(1 + \xi_1^2 + \cdots + \xi_N^2)^2 + (\xi_1^2 + \cdots + \xi_N^2)^2 - \xi_1^4 - \cdots - \xi_N^4] \tag{14}$$

The Jacobian for the change of variables (13) is proportional to Φ_{N+1}^j where

$$j = -3 + N \tag{15}$$

Thus the Jacobian will be a constant if $N = 3$. If the Jacobian is unimportant the integration over the ξ_{N+1} variable in Eq. (11) will be trivial, and will yield a delta function constraint:

$$\delta[(1 + \xi_1^2 + \cdots + \xi_N^2)^2 + (\xi_1^2 + \cdots + \xi_N^2)^2 - \xi_1^4 - \cdots - \xi_N^4].$$

Interpreting the scalar component of the superfields as coordinates, we thus see that the complex colors are constrained to lie on a hypersurface defined by the vanishing of the first and third terms of Eq. (5). In other words, if the Jacobian for the change of variables (13) is a constant, functional integration leads us back to solutions for the map coloring problem corresponding to zeros of $E(\eta)$.

CONCLUSION

The results of the last section, together with the equivalence between 2-dimensional supersymmetric quantum field theories and classical stochastic

equations noted in the introduction, suggest a connection between map coloring and Calabi-Yau non-linear σ-models. Unfortunately, the kinetic energy term in non-linear σ-models corresponds to a ferromagnetic interaction between spins [15]. Thus a discrete version of a non-linear σ-model would be fundamentally different from the map coloring algorithm whose essence is contained in the antiferromagnetic interaction between color vectors (i.e. the β-term on the r.h.s. of Eq. (5)). It is known, however, that the sign of the kinetic energy term in discrete versions of 2-dimensional σ-models can be reversed if gauge fields are added to the models [16]. This suggests that a physical analogue of the map coloring algorithm must involve world sheet gauge fields coupled to world sheet scalar or spinor fields. Candidate theories of this type have in fact been constructed to represent self-dual vortices [17] or the world sheet action for $N = 2$ superstrings [18].

In summary, we have found an interesting connection between an algorithm for coloring maps and Calabi-Yau manifolds. This connection may provide a clue as to how Nature picks a unique superstring vacuum. In particular the antiferromagnetic character of the interaction between color vectors for neighboring regions suggests that world sheet vector fields may play an important role.

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REFERENCES

1. P. Candelas, G. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. B258 (1985) 46.
2. B. R. Greene, C. Vafa, and N. Warner, Harvard preprint HUTP-88/A047.
3. P. H. Damgaard and H. Hüffel, Physics Reports 152 (1987) 228.
4. H. Nicolai, Phys. Lett. B89 (1980) 341.
5. J. Atick and E. Witten, Nucl. Phys. B310 (1988) 291.
6. G. Chapline and F. R. Klinkhamer, Mod. Phys. Lett., to appear.
7. J. P. Bouchaud and P. LeDoussal, Europhys. Lett. 1 (1986) 91.
8. E. Witten, IAS preprint HEP-88/33.
9. V. F. R. Jones, Bull. Am. Math. Soc. 12 (1985) 103.
10. L. H. Kauffman, Contemporary Math. 78 (1988) 263.
11. E. D. Dahl, Neural Network for an NP-Complete Problem, Proceedings of the International Conference on Neural Networks, San Diego (IEEE, 1987).
12. J. J. Hopfield, Proc. Natl. Acad. Sci. 79 (1982) 2554.
13. T. L. Saaty and P. C. Kainen, The Four-Color Problem, Dover (1986).
14. G. Ringel and J. Youngs, Proc. Nat. Acad. Sci. U.S.A. 60 (1968) 438.
15. R. Balian, J. M. Droutle, and C. Itzykson, Phys. Rev. D10 (1974) 3376.
16. V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. 59 (1987) 2095.
17. G. Chapline and B. Grossmann, Phys. Lett. to be published.
18. G. Chapline, Int. J. Mod. Phys. 3 (1988) 1663.