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PIECEWISE CUBIC INTERPOLATION METHODS

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November 1978

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# PIECEWISE CUBIC INTERPOLATION METHODS

F.N. Fritsch, R.E. Carlson[1]

November 1978

## A B S T R A C T

This paper deals with interpolation of one-dimensional data using piecewise cubic interpolants. Methods are presented for modifying the derivative values in the Hermite representation in order to eliminate the "bumps" and "wiggles" that frequently plague the more common cubic spline or Akima interpolants. The resulting interpolant is $C^1$, but generally not $C^2$.

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*Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore Laboratory under contract number W-7405-ENG-48.

This report consists of a reproduction of a poster prepared for the SIAM 1978 Fall Meeting. A more complete description of our new algorithm is being prepared for publication.

The poster contained a two-dimensional display of methods vs data sets to facilitate comparison of the six methods on four sets of data. This is simulated here by numbering the figures i.j. as follows:

Data Sets:

i = 1. LLL data set RPN 12*

i = 2. LLL data set RPN 14*

i = 3. Example 3 from Akima's paper (see page 5).

i = 4. A nonmonotone example. These data are from S. Pruess, "An Algorithm for Computing Smoothing Splines in Tension", Computing 19 (1978), 365-373.

Methods:

j = 1. Cubic Splines.

j = 2. 3-Point Difference Formula.

j = 3. Ellis-McLain Method.

j = 4. Akima Method.

j = 5. Zero Derivatives.

j = 6. A new method by the authors that guarantees a monotone interpolant when the data are monotone.

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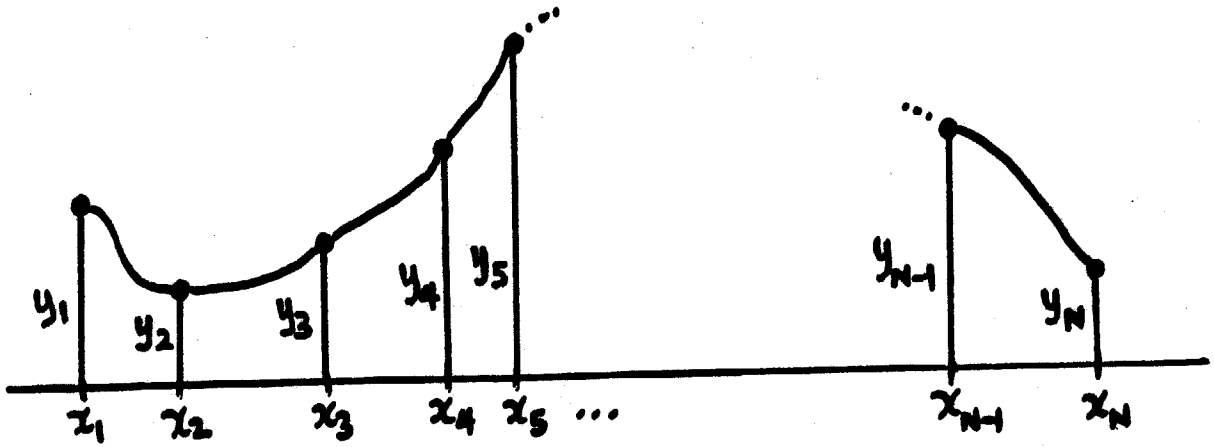
*Actual data from a radiochemical calculation.

# PIECEWISE CUBIC INTERPOLANT




$f(x) = c_i(x), \quad x_i \le x \le x_{i+1}$ (cubic polynomial)

$c_i(x_i) = y_i, \quad c_i(x_{i+1}) = y_{i+1}$ $(i = 1, \dots, N-1)$

$c_i'(x_{i+1}) = d_{i+1} = c_{i+1}'(x_{i+1})$ $(i = 1, \dots, N-2)$

# Cubic Spline

- $f(x) \in \mathcal{C}^2[x_1, x_N]$; $f(x_i) = y_i$, $i = 1, \ldots, N$.
- Two degrees of freedom, generally used to specify endpoint first or second derivative values.
- This version uses 3-point (non-centered) difference formulas to approximate end derivatives.
- Interpolants can have unphysical wiggles.

(See Figures i.1)

# 3-Point Difference

- $d_i$ is the derivative at $x_i$ of the quadratic that passes through $(x_{i-1}, y_{i-1})$, $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$.

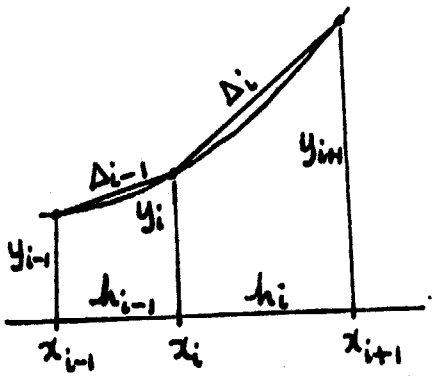
- $d_i$ is a convex combination of the slopes of the adjacent data:

$$d_i = \frac{h_i}{h_{i-1}+h_i}\Delta_{i-1} + \frac{h_{i-1}}{h_{i-1}+h_i}\Delta_i$$

where

$$h_j = x_{j+1} - x_j ,$$

$$\Delta_j = (y_{j+1} - y_j)/h_j .$$

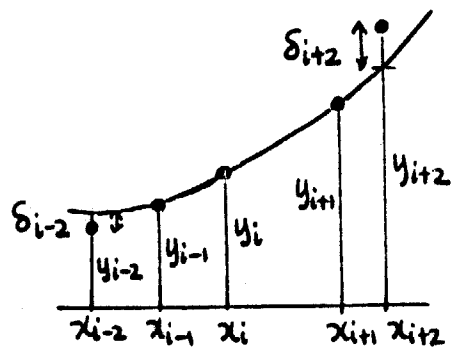



(See Figures i.2)

# Ellis-McLain

- $d_i$ is the derivative at $x_i$ of the cubic that passes through $(x_{i-1}, y_{i-1})$, $(x_i, y_i)$, $(x_{i+1}, y_{i+1})$ and provides best (weighted) least squares fit to $(x_{i-2}, y_{i-2})$ and $(x_{i+2}, y_{i+2})$:

$$\min \left( \frac{\delta_{i-2}^2}{h_{i-2}^2} + \frac{\delta_{i+2}^2}{h_{i+1}^2} \right).$$




- Results are generally quite similar to 3PD.
- Reference: T.M.R. Ellis and D.H. McLain, "Algorithm 514, A New Method of Cubic Curve Fitting Using Local Data," ACM-TOMS 3 (1977), 175-178.

(See Figures i.3)

# Akima

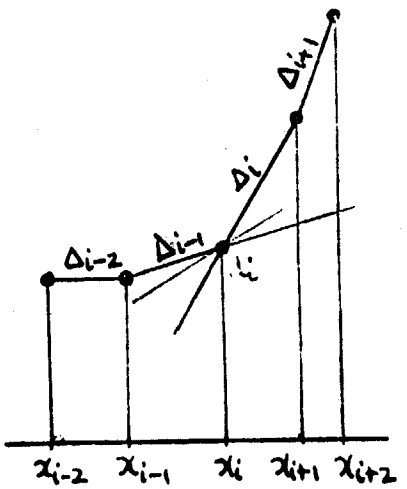
- $d_i$ is a convex combination of the slopes of the adjacent data, derived by a geometric argument:

$$d_i = \frac{a_i}{a_i + b_i}\Delta_{i-1} + \frac{b_i}{a_i + b_i}\Delta_i ,$$

where

$$a_i = |\Delta_{i+1} - \Delta_i| \quad , \quad b_i = |\Delta_{i-1} - \Delta_{i-2}| \quad .$$

- Reference: H. Akima, "A New Method of Interpolation and Smooth Curve Fitting Based on Local Procedures," J. ACM 17 (1970), 589-602.




(See Figures i.4)

# Zero Derivatives

- The simplest possible algorithm is to just set $d_i = 0$, $i = 1, \ldots, N$. As noted in the references, this turns out to be piecewise monotone.

- As the examples illustrate, this produces interpolants that are extremely "unphysical". Thus, while being monotone where the data are may be necessary, it is not sufficient to produce interpolants that "look good".

- References: (1) E. Passow, "Piecewise Monotone Spline Interpolation," J. Approx. Theory 12 (1974), 240-241.
  (2) C. deBoor and B. Swartz, "Piecewise Monotone Interpolation," J. Approx. Theory 21 (1977), 411-416.

(See Figures i.5)

# Fritsch-Carlson

Toward a piecewise monotone interpolant that "looks good".

- $\Delta_i := (y_{i+1} - y_i)/h_i$, $h_i := x_{i+1} - x_i$. If $\Delta_i = 0$, $f(x)$ cannot be monotone on $[x_i, x_{i+1}]$ unless $d_i = d_{i+1} = 0$.
- Suppose $\Delta_i \neq 0$ and let $\alpha = d_i/\Delta_i$, $\beta = d_{i+1}/\Delta_i$. Then

$$c_i(x) = \frac{\Delta_i}{h_i^2}\Big[(\alpha+\beta-2)(x-x_i)^3 - (2\alpha+\beta-3)h_i(x-x_i)^2 + \alpha h_i^2(x-x_i)\Big] + y_i,$$

$$c_i'(x) = \frac{\Delta_i}{h_i^2}\Big[3(\alpha+\beta-2)(x-x_i)^2 - 2(2\alpha+\beta-3)h_i(x-x_i)$$

$$+ \alpha h_i^2 \Big],$$

$$c_i''(x) = \frac{2\Delta_i}{h_i^2}\left[3(\alpha+\beta-2)(x-x_i) - (2\alpha+\beta-3)h_i\right].$$

- It is easy to show that a necessary condition for monotonicity of $c_i(x)$ on $[x_i, x_{i+1}]$ is $\alpha \geq 0$, $\beta \geq 0$.

- Because $c_i'(x)$ is quadratic, monotonicity of $c_i(x)$ is directly related to the location of the extremum of $c_i'$:

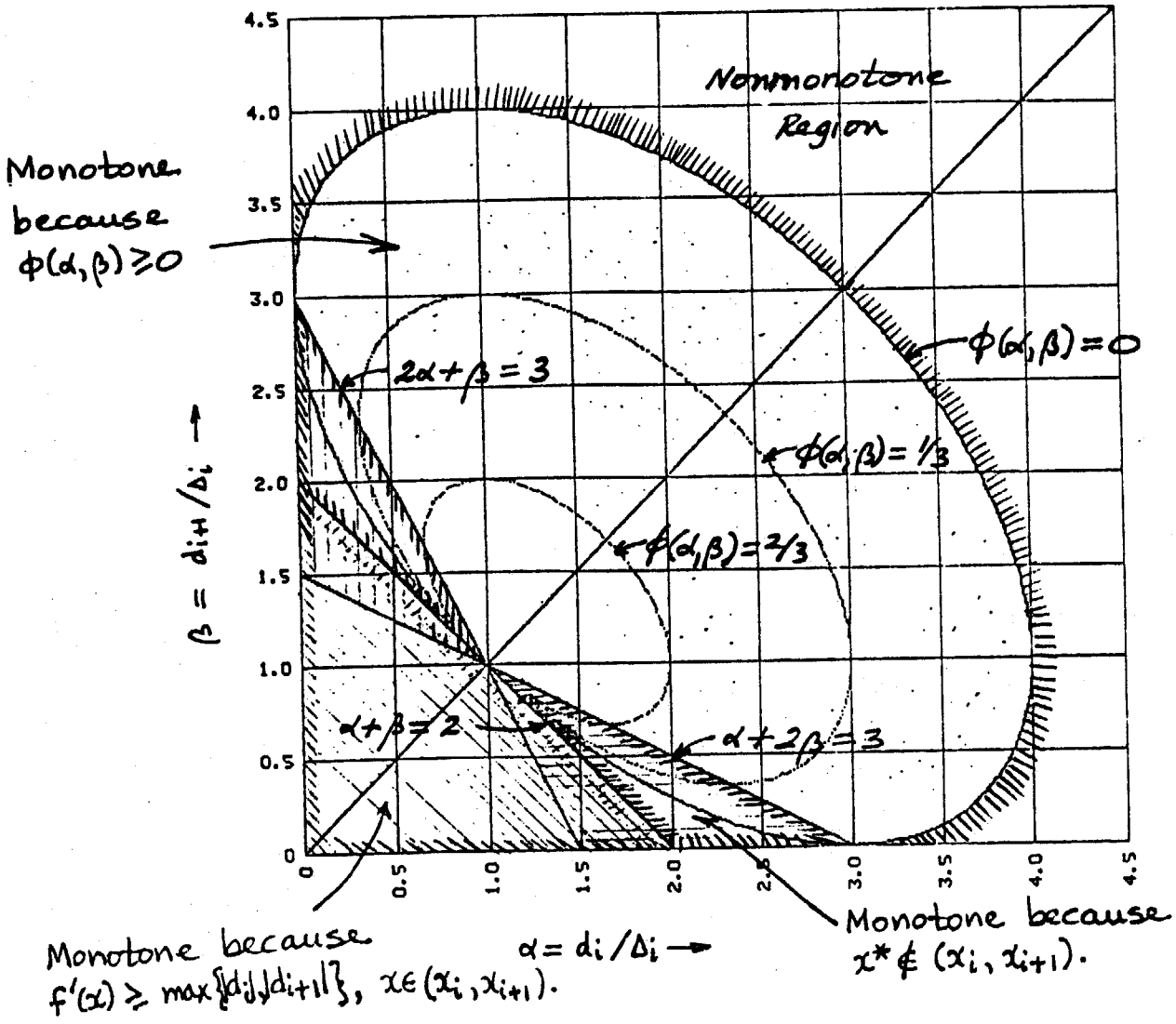
$$x^* = x_i + \frac{h_i}{3}\left(\frac{2\alpha+\beta-3}{\alpha+\beta-2}\right) \qquad (\alpha+\beta \neq 2)$$

and its value there:

$$c_i'(x^*) = \phi(\alpha,\beta)\cdot\Delta_i,$$

$$\phi(\alpha,\beta) = \alpha - \frac{1}{3}\frac{(2\alpha+\beta-3)^2}{\alpha+\beta-2}.$$

- Thus, we can characterize the monotonicity properties of $f(x)$ on $[x_i, x_{i+1}]$ by the following diagram.


Nonmonotone Region
Monotone because $\phi(\alpha,\beta) \geq 0$
$\phi(\alpha,\beta) = 0$
$2\alpha + \beta = 3$
$\phi(\alpha,\beta) = 1/3$
$\phi(\alpha,\beta) = 2/3$
$\alpha + \beta = 2$
$\alpha + 2\beta = 3$
$\beta = d_{i+1}/\Delta_i \rightarrow$
$\alpha = d_i/\Delta_i \rightarrow$
Monotone because $f'(x) \geq \max\{|d_i|, |d_{i+1}|\}$, $x \in (x_i, x_{i+1})$.
Monotone because $x^* \notin (x_i, x_{i+1})$.

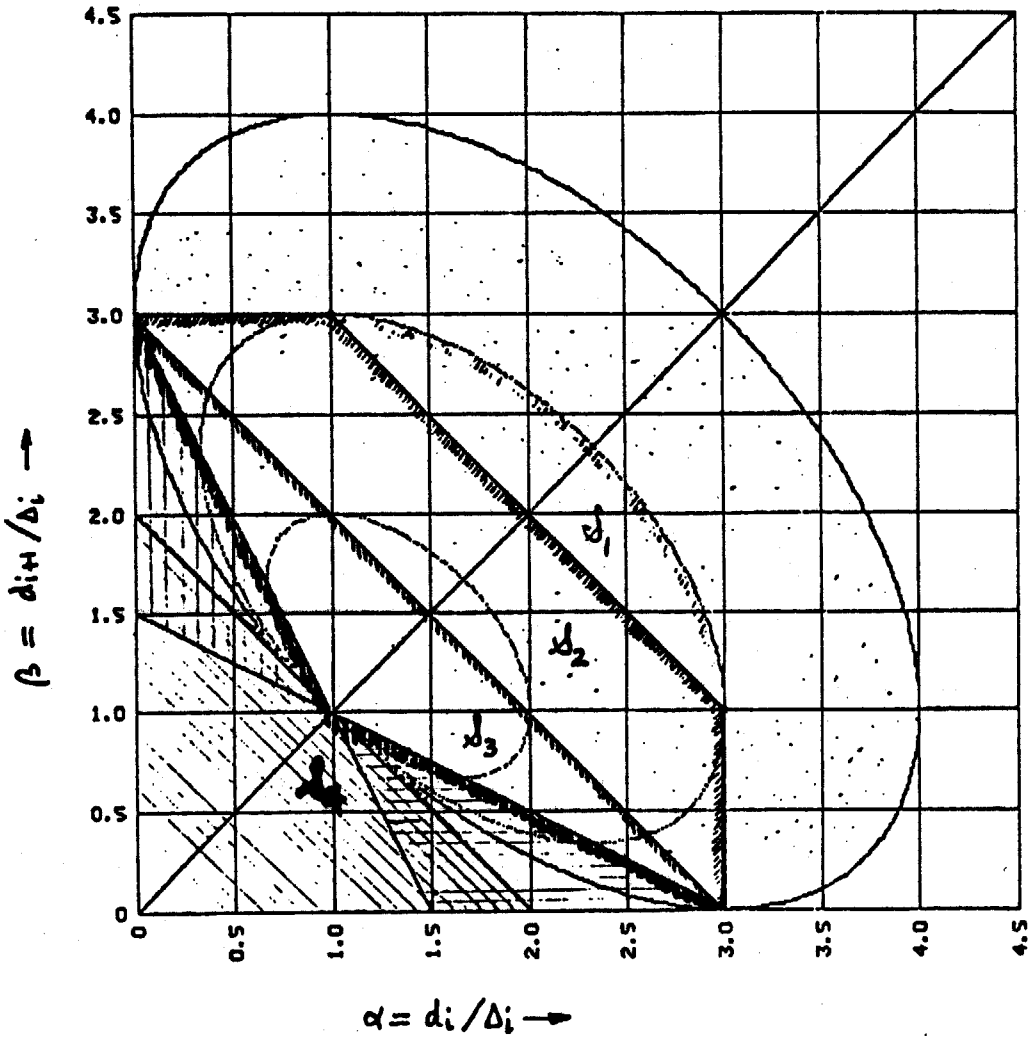
- Let $\mathcal{M}$ be the region bounded by $\phi(\alpha,\beta)=0$ and the coordinate axes. This is the set of monotone interpolants on $[x_i, x_{i+1}]$.

- Let $\mathcal{S}$ be a subset of $\mathcal{M}$ with the property:
  Let $(\alpha,\beta)\in\mathcal{S}$. (i) If $0\le\alpha^*\le\alpha$, $0\le\beta^*\le\beta$, then $(\alpha^*,\beta^*)\in\mathcal{S}$. (ii) $(\beta,\alpha)\in\mathcal{S}$ (symmetry).
  If we adjust $(\alpha,\beta)$ to lie inside $\mathcal{S}$ by decreasing $\alpha$ and/or $\beta$, (i) insures that we do not destroy monotonicity in an adjacent interval.

- Regions $\mathcal{S}$ we have considered:

  $\mathcal{S}_1$ bounded by $\alpha=3$, $\beta=3$, and $\phi(\alpha,\beta)=1/3$.
  $\mathcal{S}_2$ bounded by $\alpha=3$, $\beta=3$, and $\alpha+\beta=4$.
  $\mathcal{S}_3$ bounded by $\alpha+\beta=3$.
  $\mathcal{S}_4$ bounded by $2\alpha+\beta=3$ and $\alpha+2\beta=3$.
  (All bounded below by coordinate axes.)

Suggested Monotonicity Subregions $\mathcal{S}_i$



- A family of piecewise monotone piecewise cubic interpolation schemes:

Step 1. Initialize $d_i$ to some convex combination of $\Delta_{i-1}$ and $\Delta_i$. [For these examples we use 3PD.]

Step 2a. If $\Delta_i = 0$, set $d_i = d_{i+1} = 0$. and go to next interval.

Step 2b. If $\Delta_i \neq 0$, compute $\alpha$ and $\beta$.

Step 3a. If $\alpha \geq 0$, $\beta \geq 0$ and $(\alpha, \beta) \in \mathcal{S}$, go to next interval.

Step 3b. If $\alpha \geq 0$, $\beta \geq 0$ and $(\alpha, \beta) \notin \mathcal{S}$, compute the largest $\tau$, $0 \leq \tau \leq 1$, such that $(\tau\alpha, \tau\beta) \in \mathcal{S}$. Set $d_i := \tau d_i$, $d_{i+1} := \tau d_{i+1}$. [Here we use $\mathcal{S} = \mathcal{S}_3$.]

Step 3c. If $\alpha < 0$ or $\beta < 0$ the data are nonmonotone. Possible procedures:

(a) Leave $d_i$, $d_{i+1}$ unchanged and go to next interval.

(b) Set $d_i = d_{i+1} = 0$. [This insures exact piecwise monotonicity.]

[Procedure (a) was used for these examples.]

(See Figures i.6)

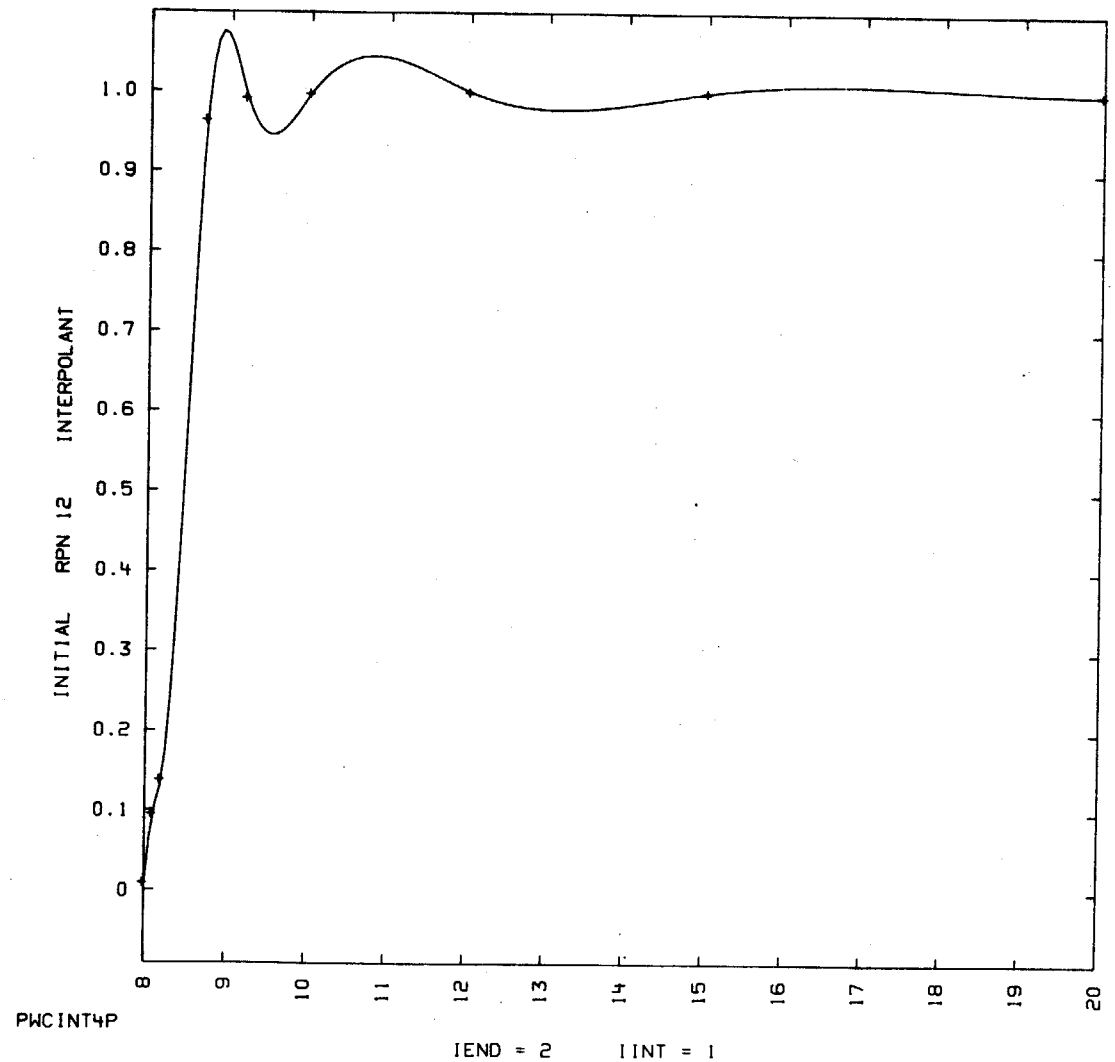



Figure 1.1. Cubic Spline on Data Set 1.

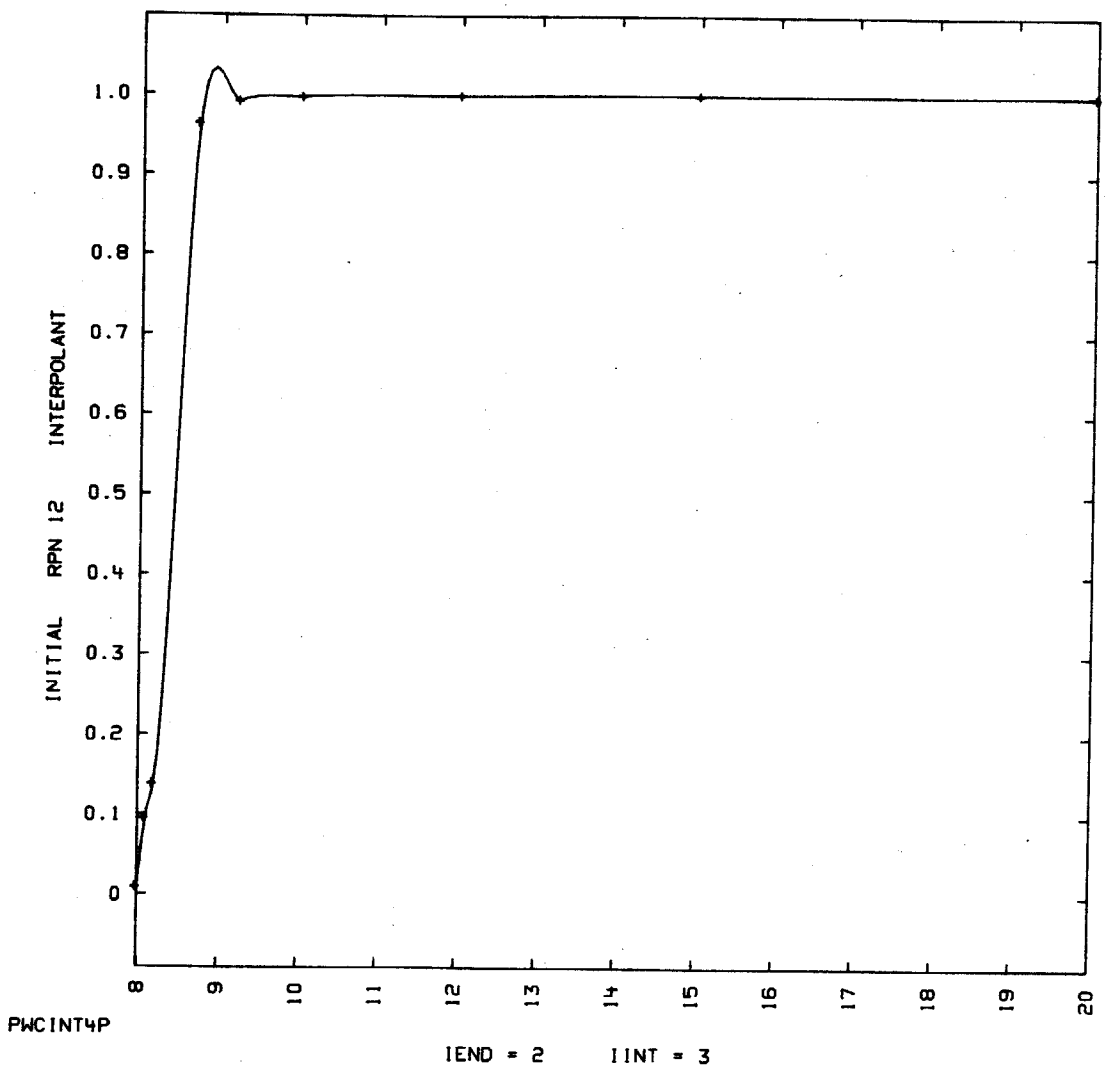



Figure 1.2. 3-Point Difference Formula on Data Set 1.

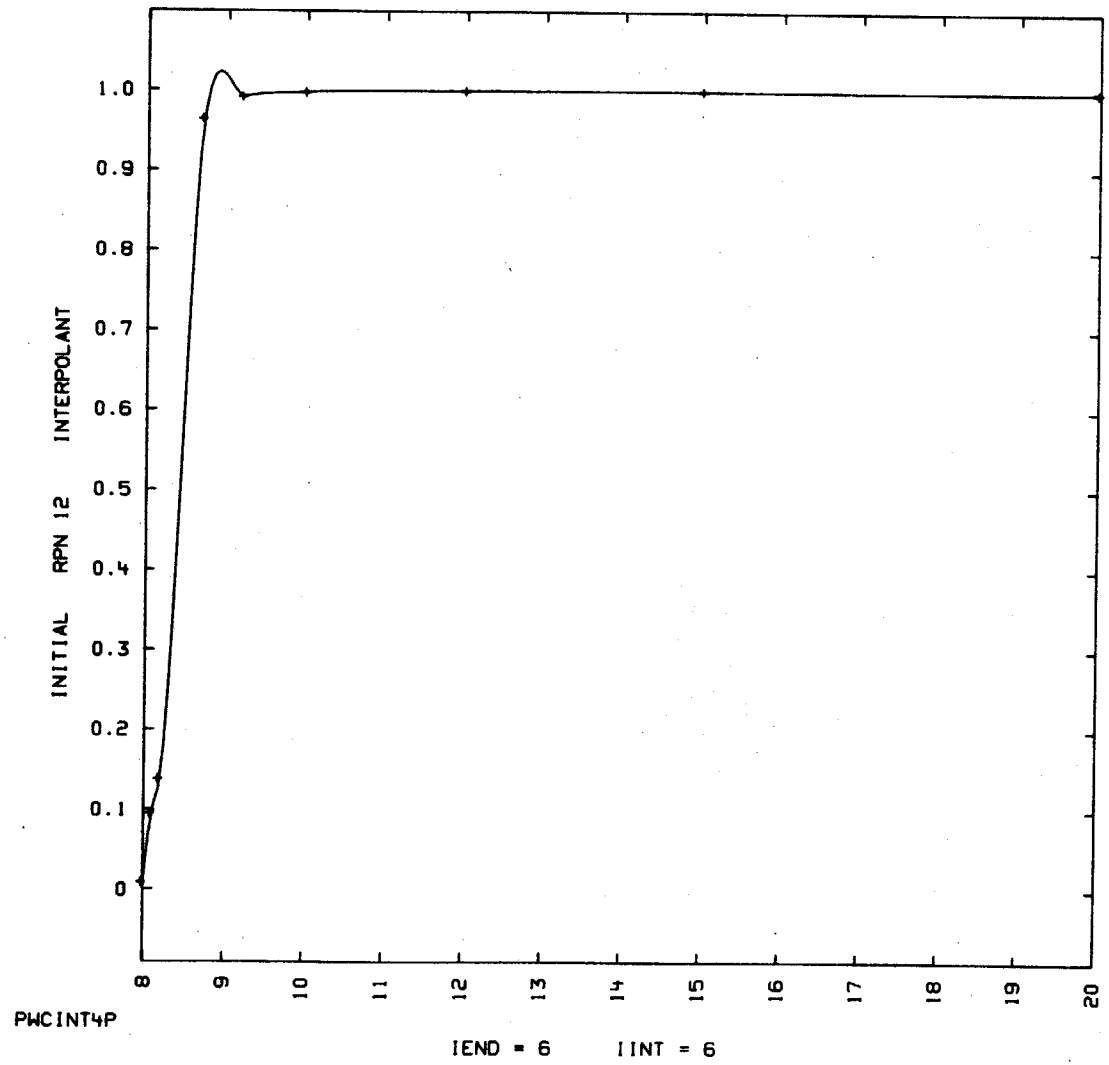



Figure 1.3. Ellis-McLain Method on Data Set 1.

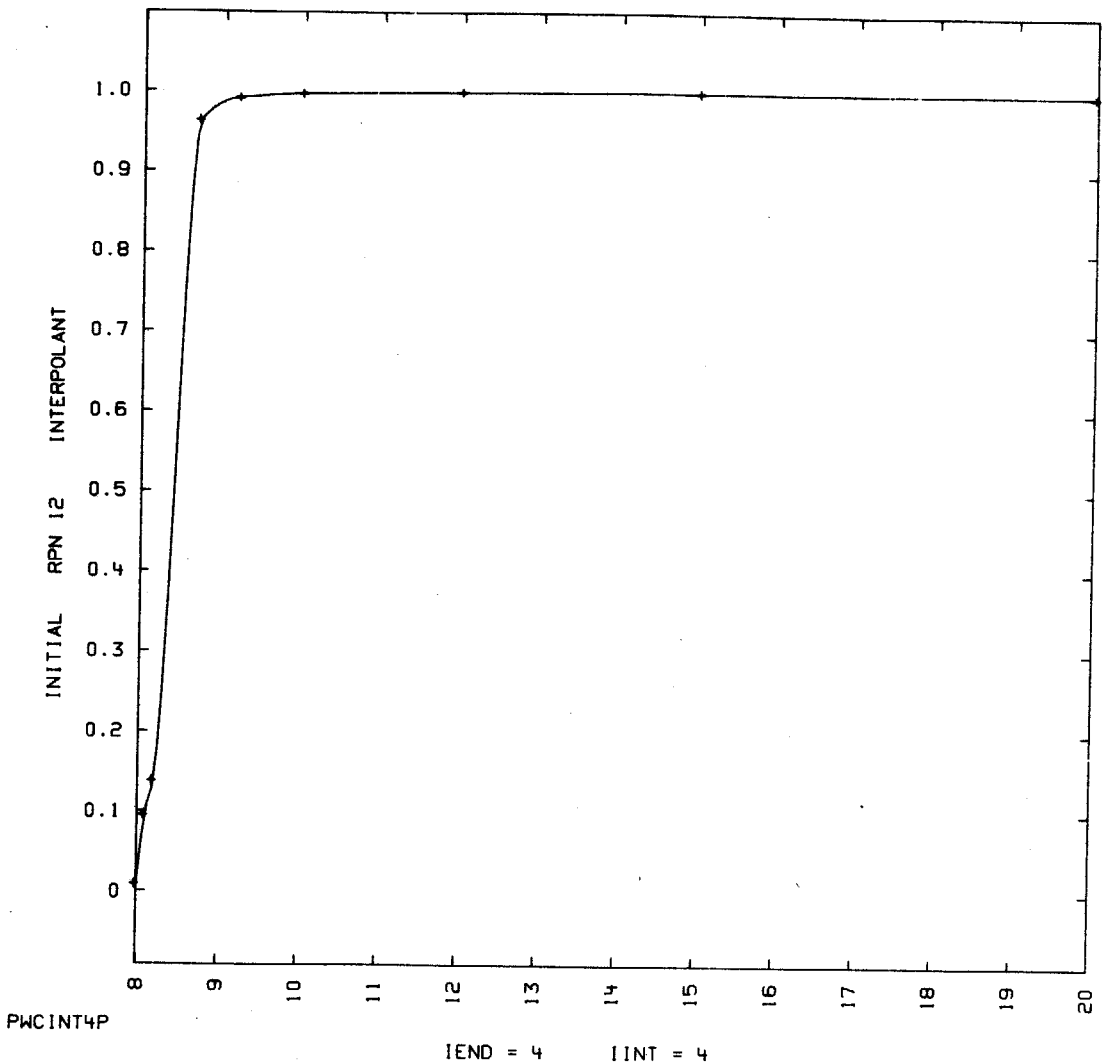



Figure 1.4. Akima Method on Data Set 1.

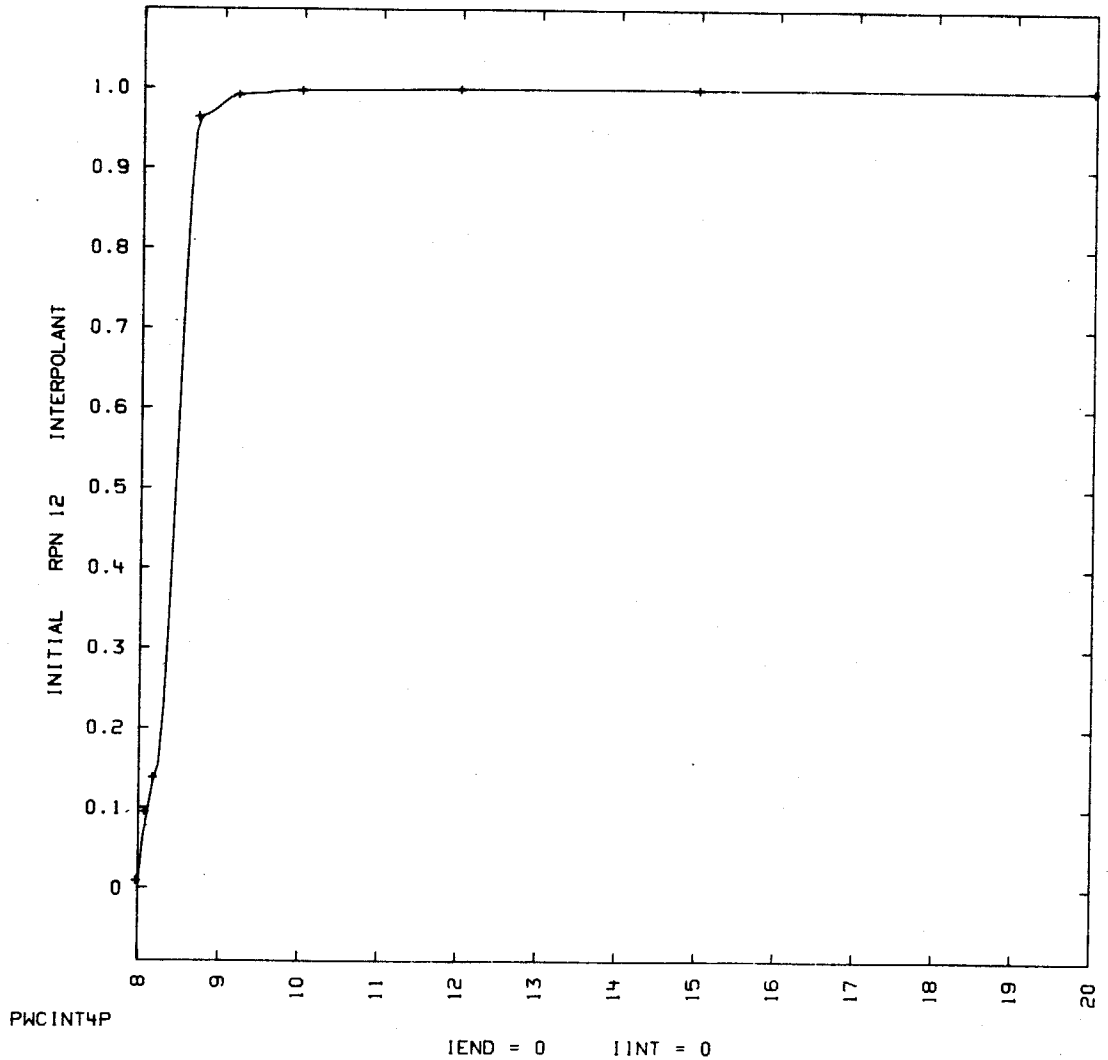



Figure 1.5. Zero Derivatives on Data Set 1.

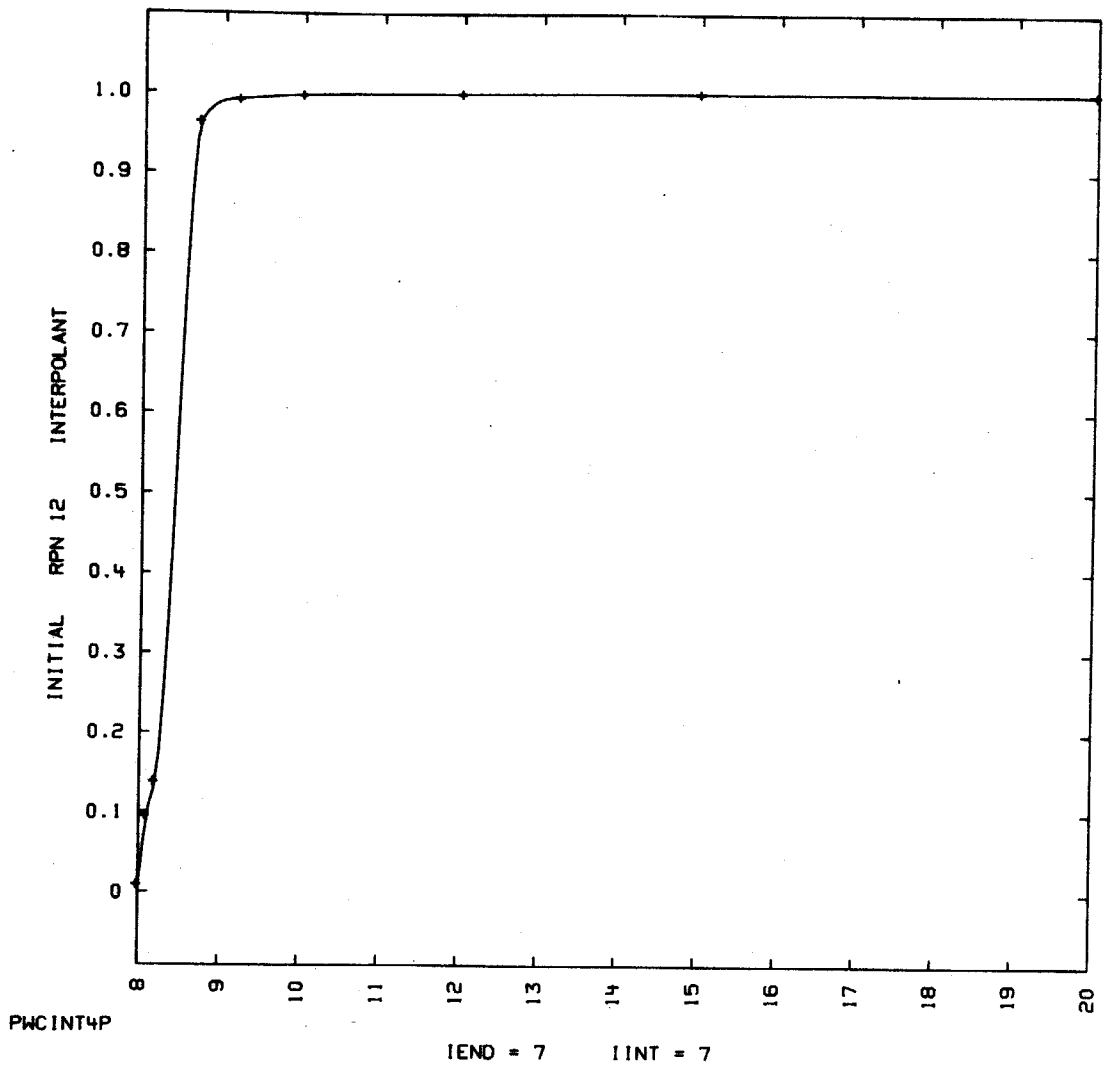



Figure 1.6. Fritsch-Carlson Method on Data Set 1.

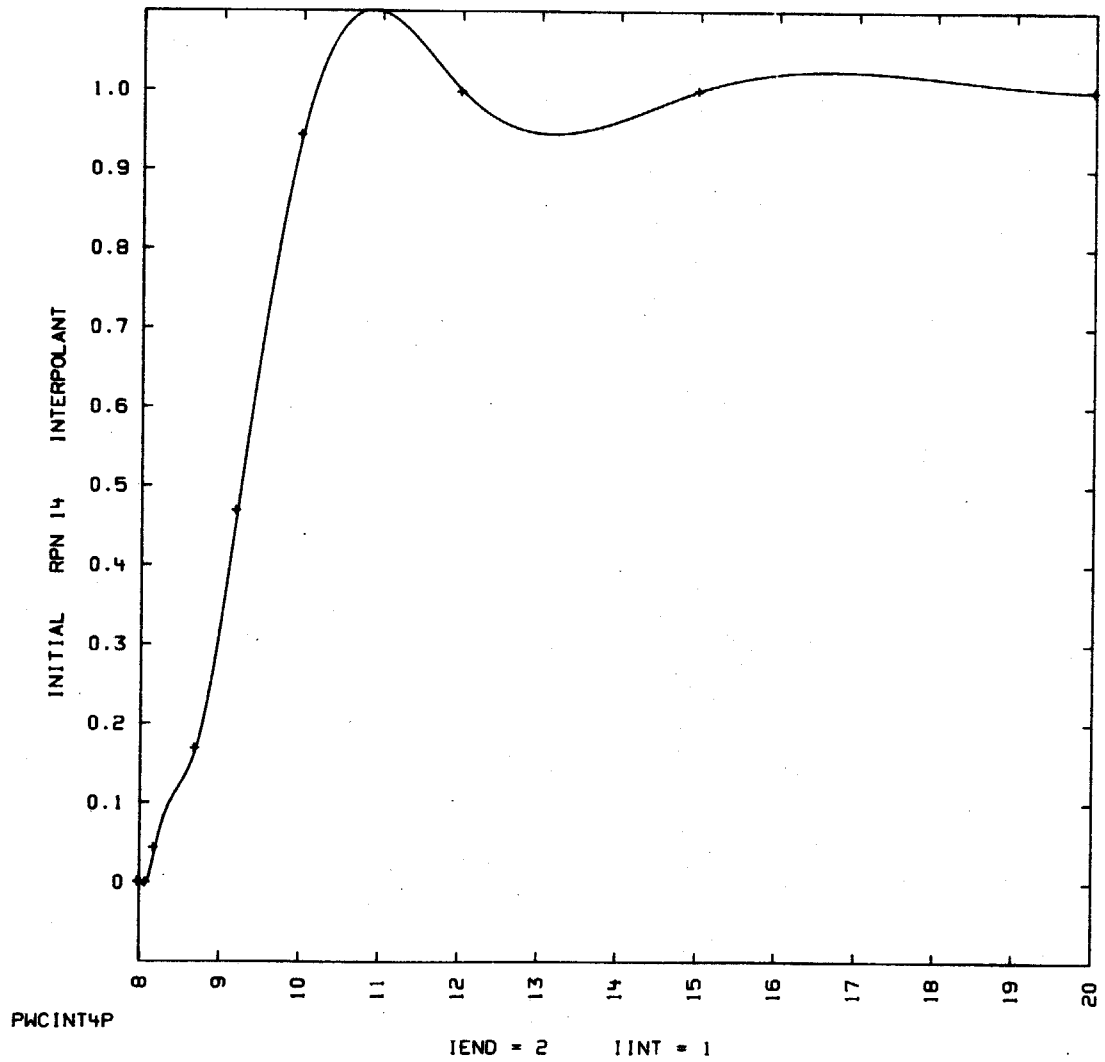



Figure 2.1. Cubic Spline on Data Set 2.

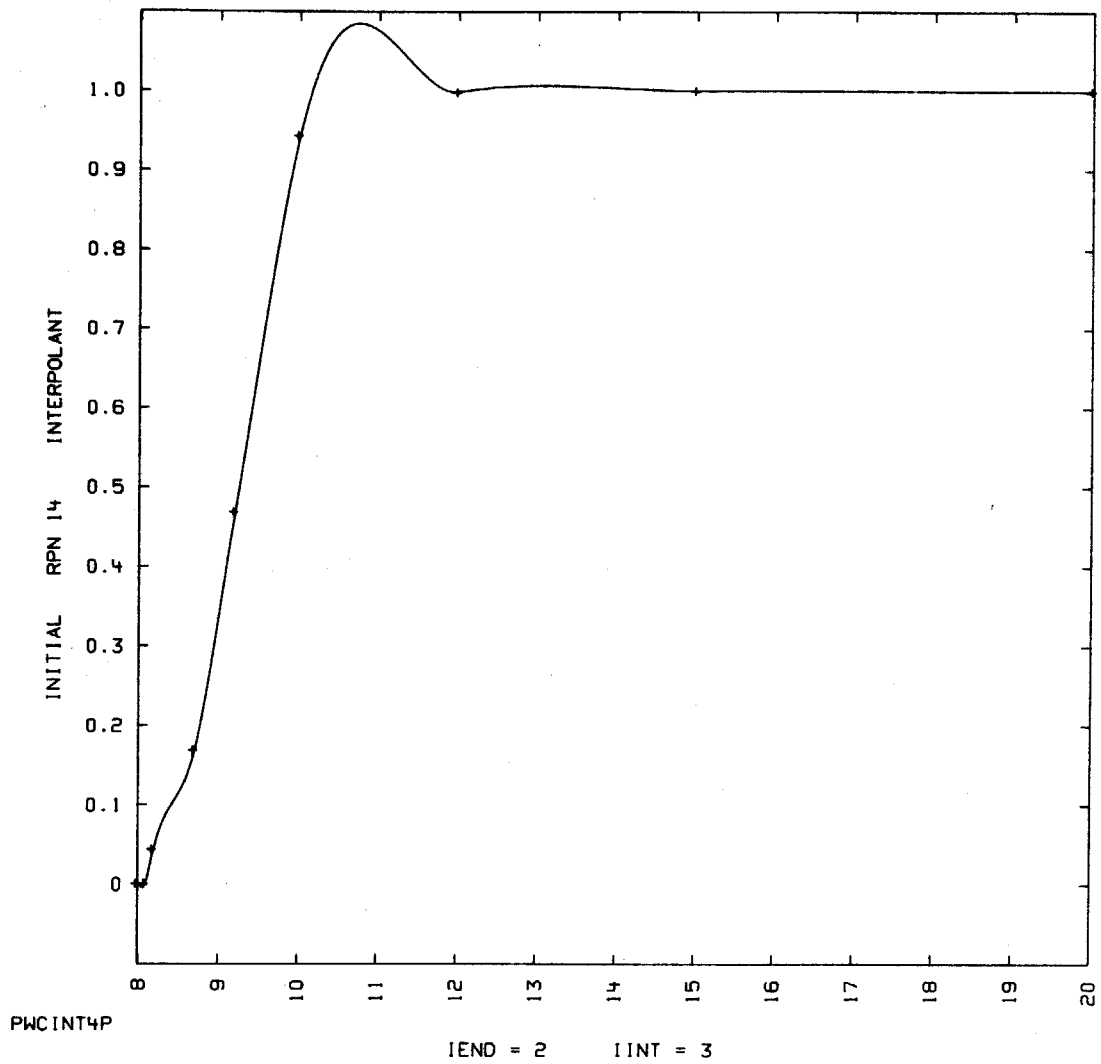



Figure 2.2. 3-Point Difference Formula on Data Set 2.

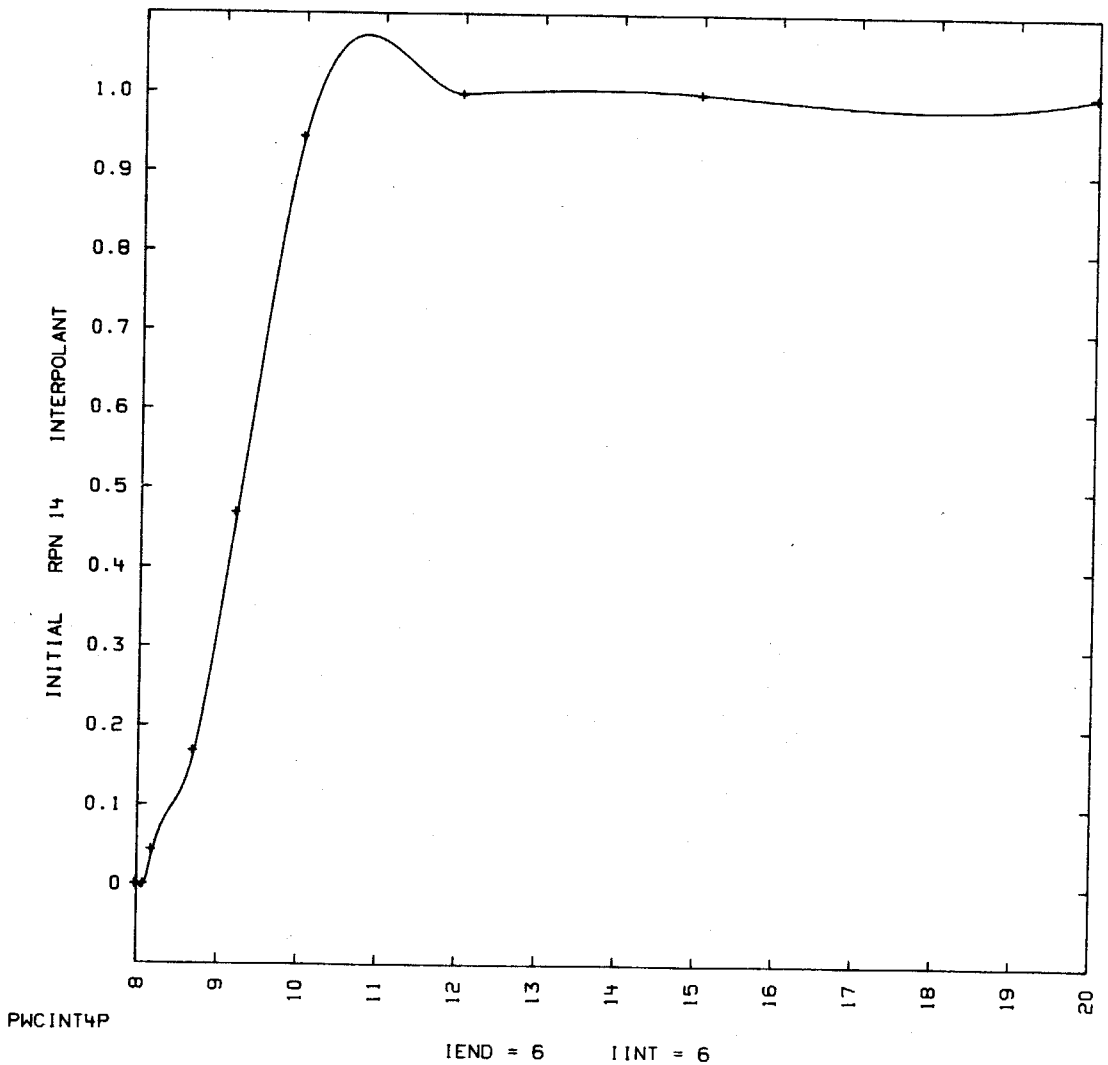



Figure 2.3. Ellis-McLain Method on Data Set 2.

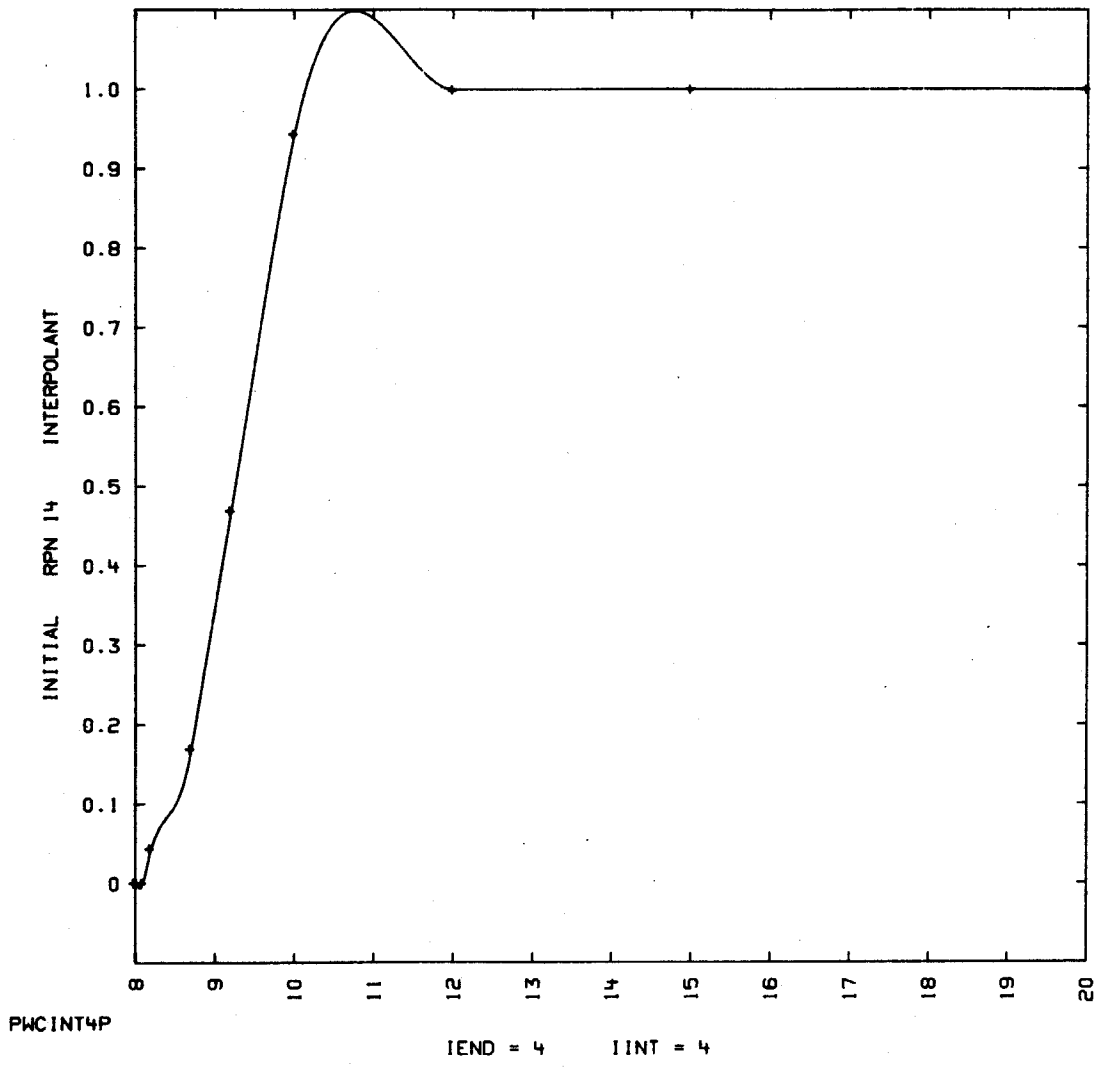



Figure 2.4. Akima Method on Data Set 2.

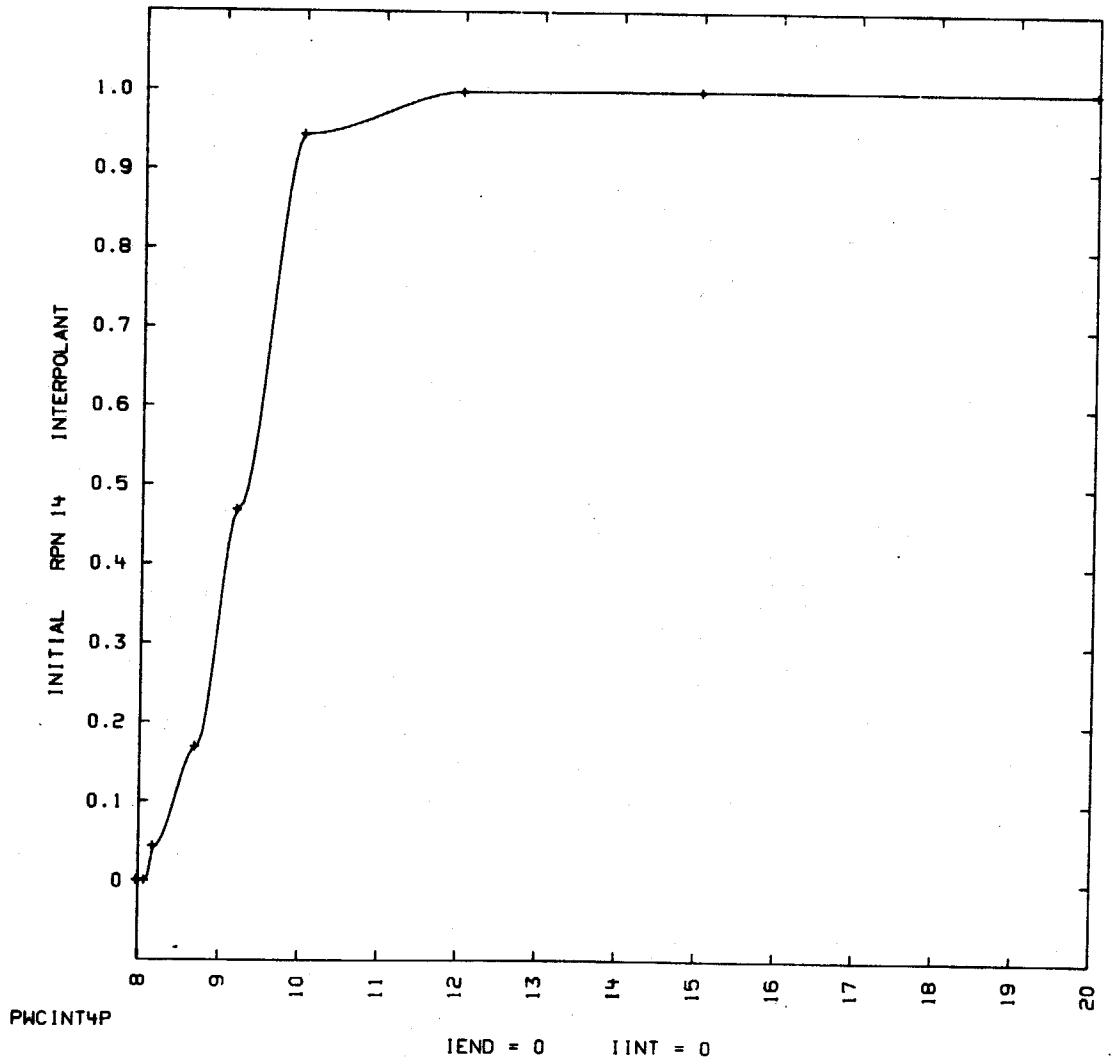



Figure 2.5. Zero Derivatives on Data Set 2.

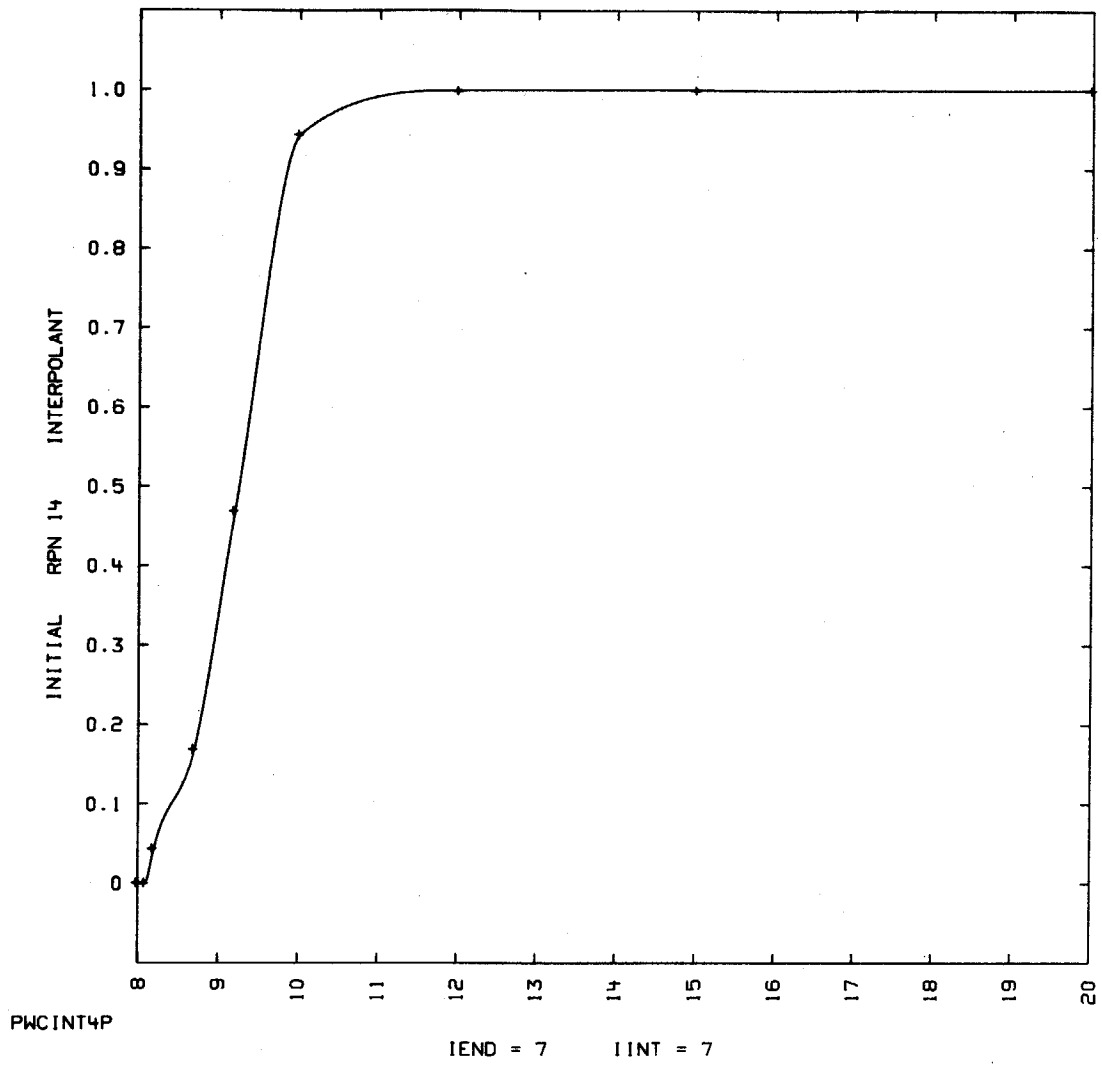



Figure 2.6. Fritsch-Carlson Method on Data Set 2.

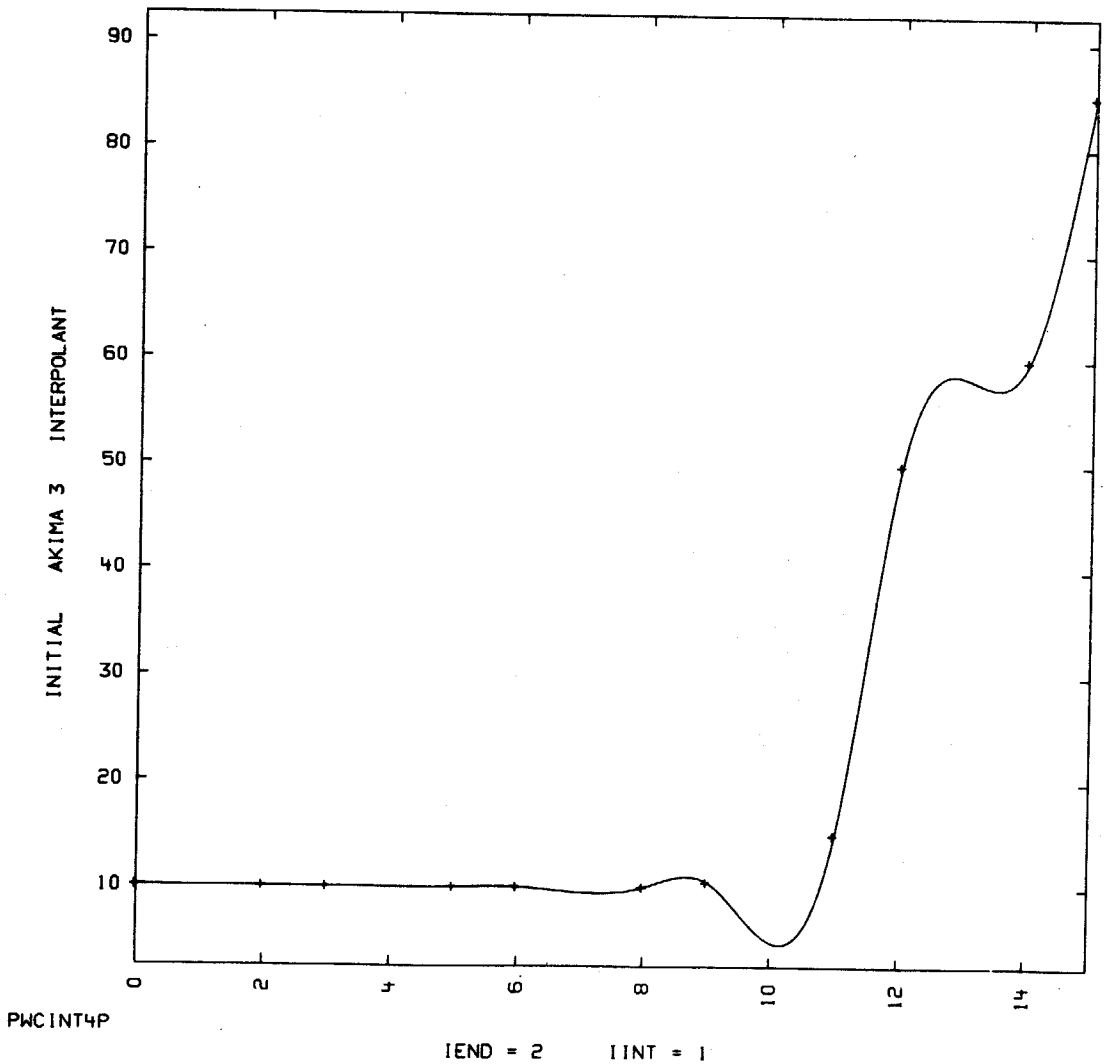



Figure 3.1. Cubic Spline on Data Set 3.

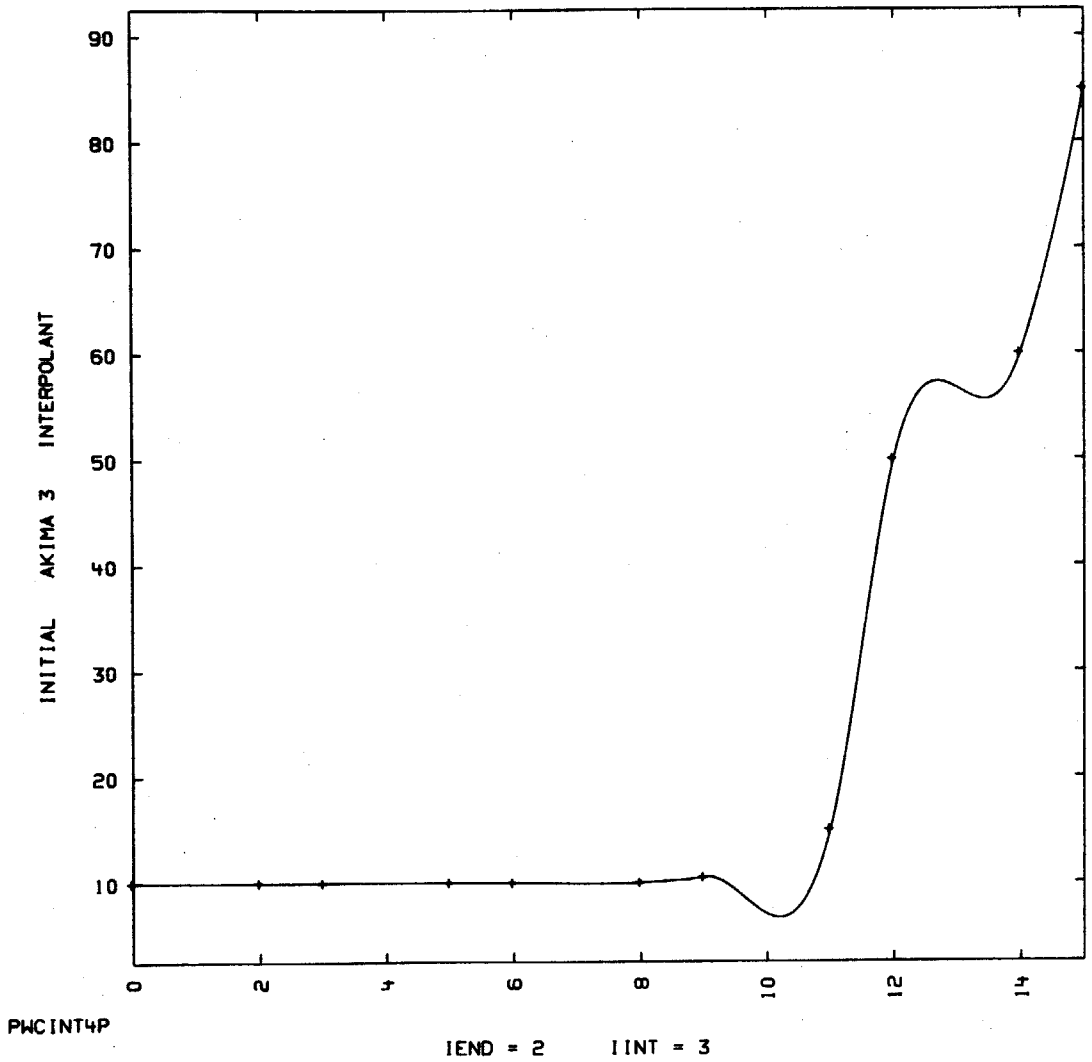



Figure 3.2. 3-Point Difference Formula on Data Set 3.

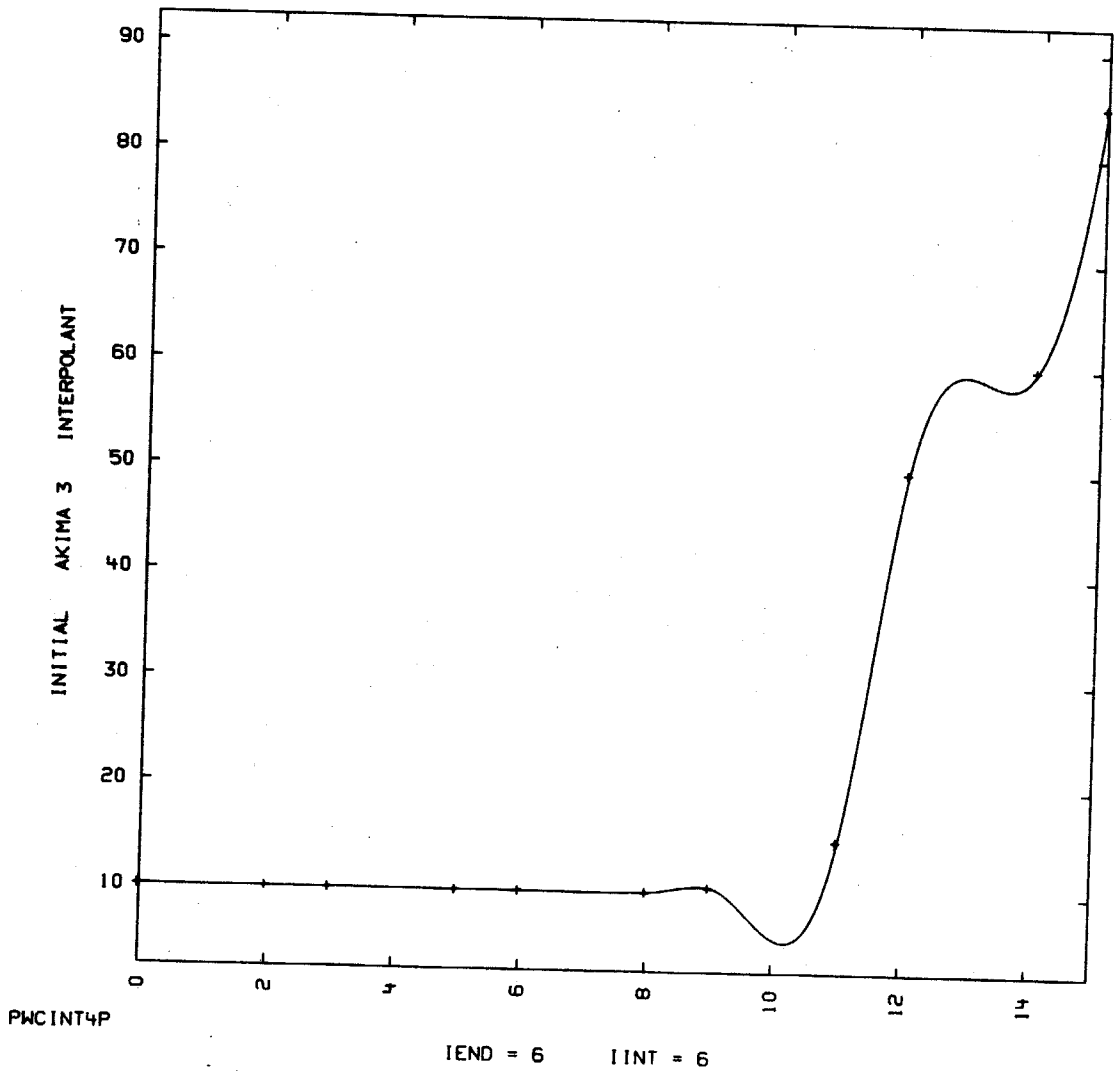



Figure 3.3. Ellis-McLain Method on Data Set 3.

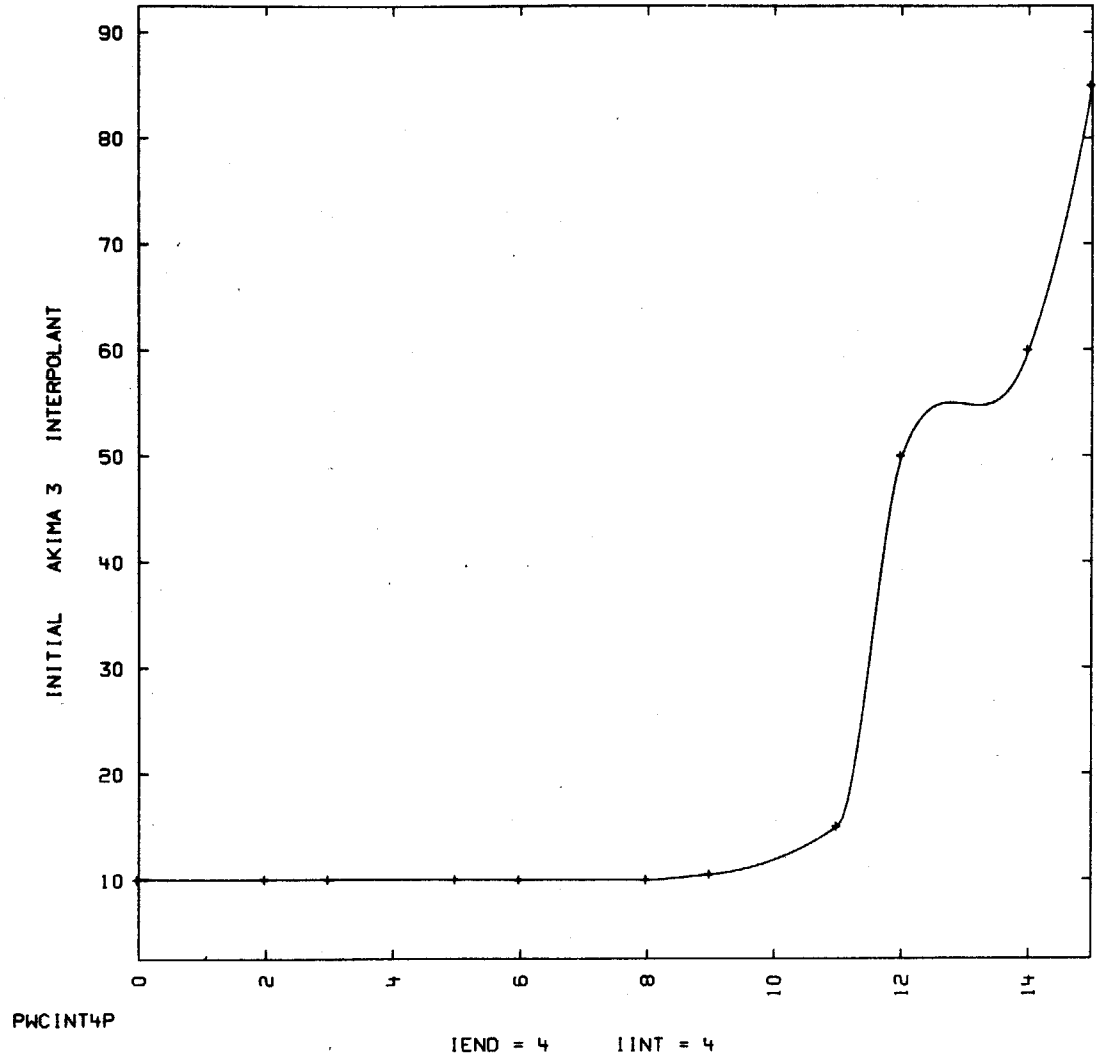



Figure 3.4. Akima Method on Data Set 3.

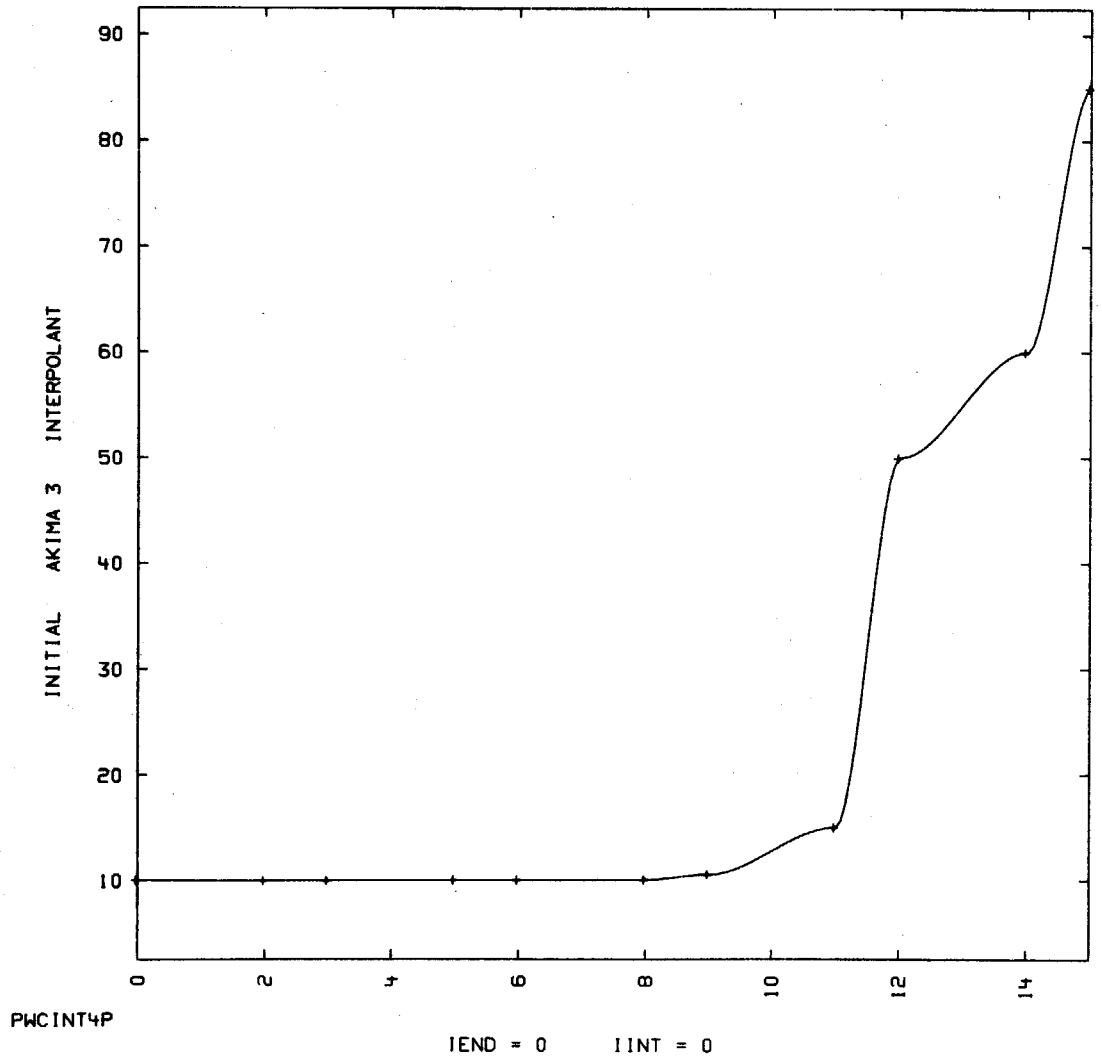



Figure 3.5. Zero Derivatives on Data Set 3.

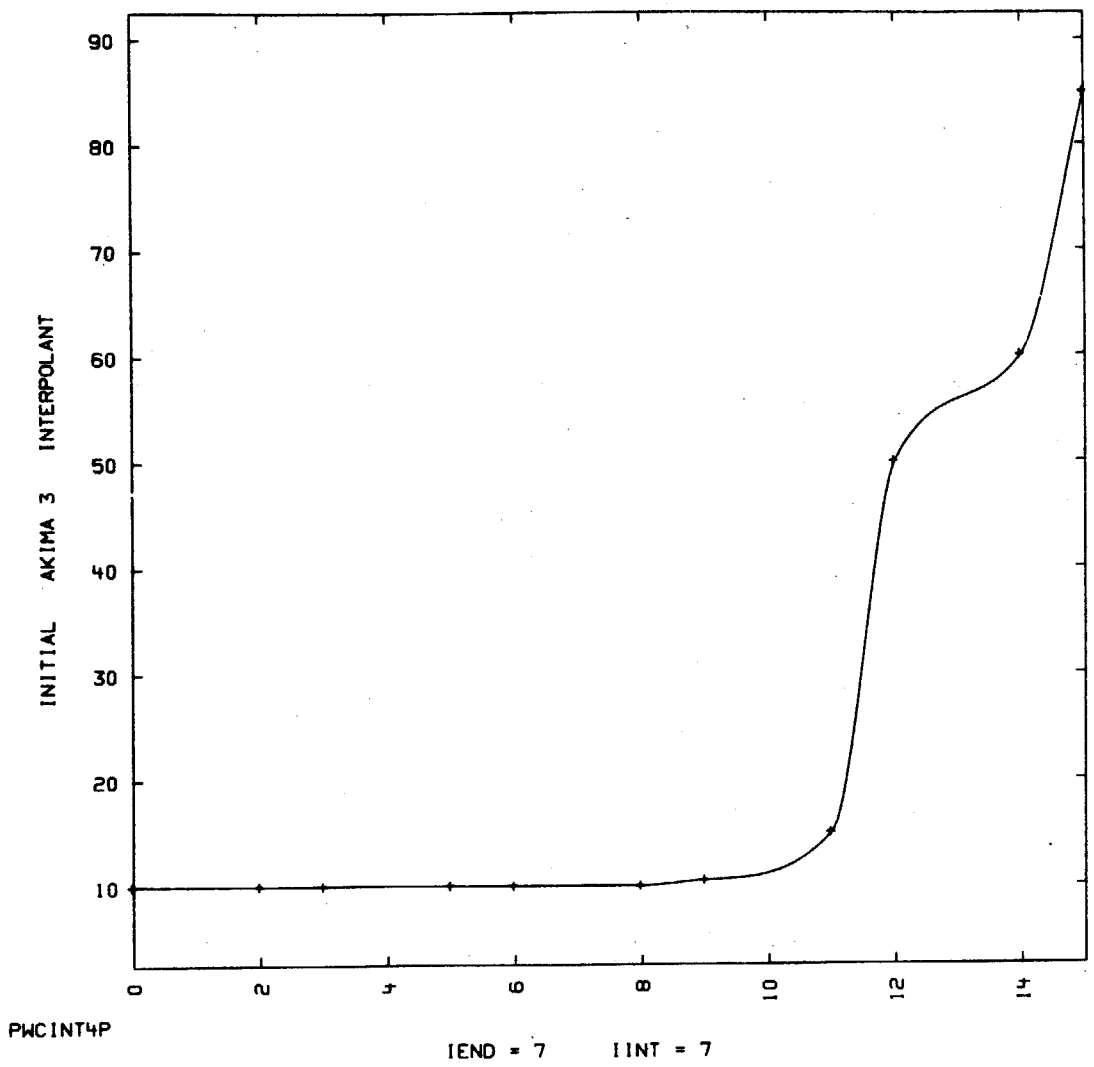



Figure 3.6. Fritsch-Carlson Method on Data Set 3.

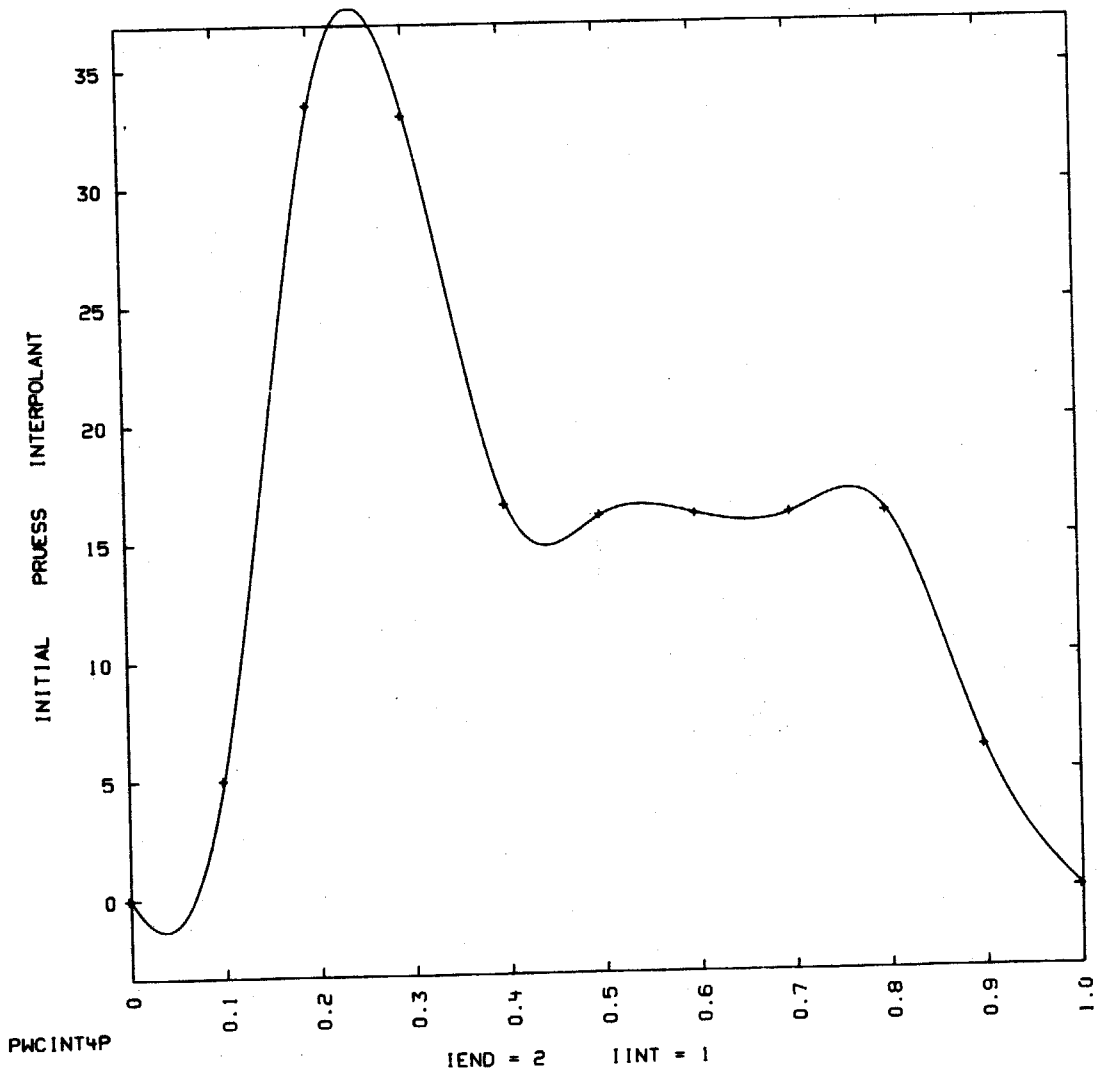



Figure 4.1. Cubic Spline on Data Set 4.

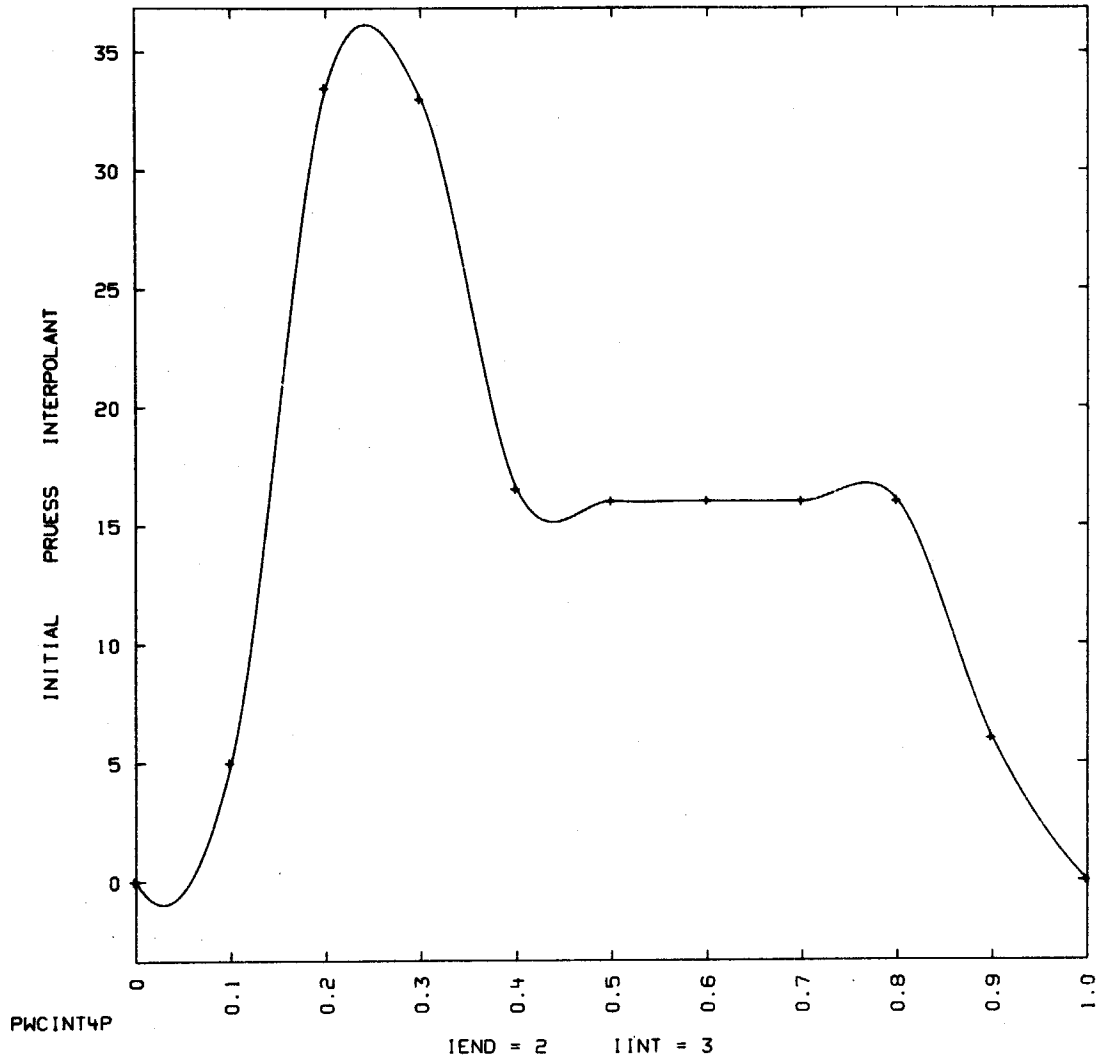



Figure 4.2. 3-Point Difference Formula on Data Set 4.

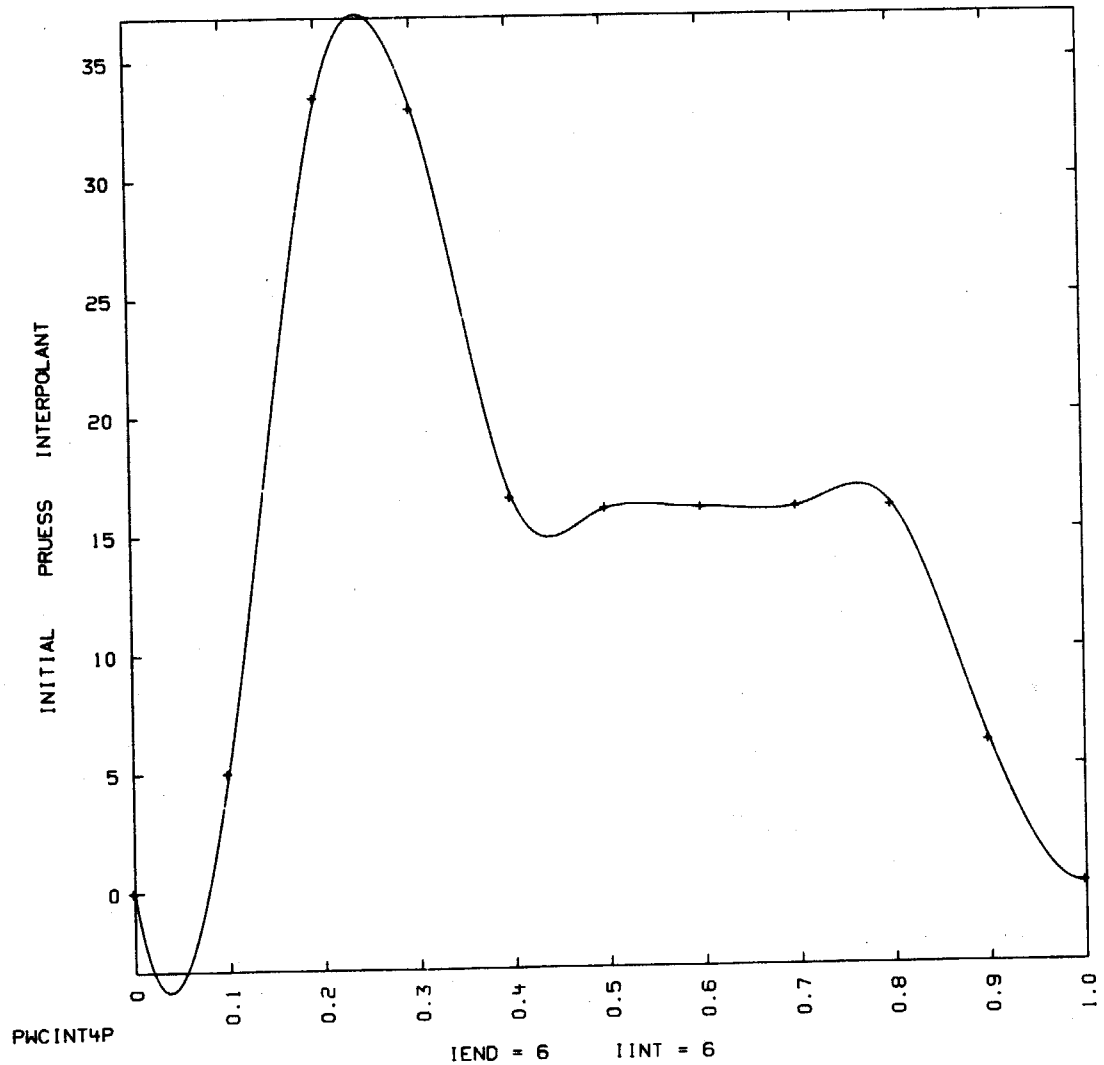



Figure 4.3. Ellis-McLain Method on Data Set 4.

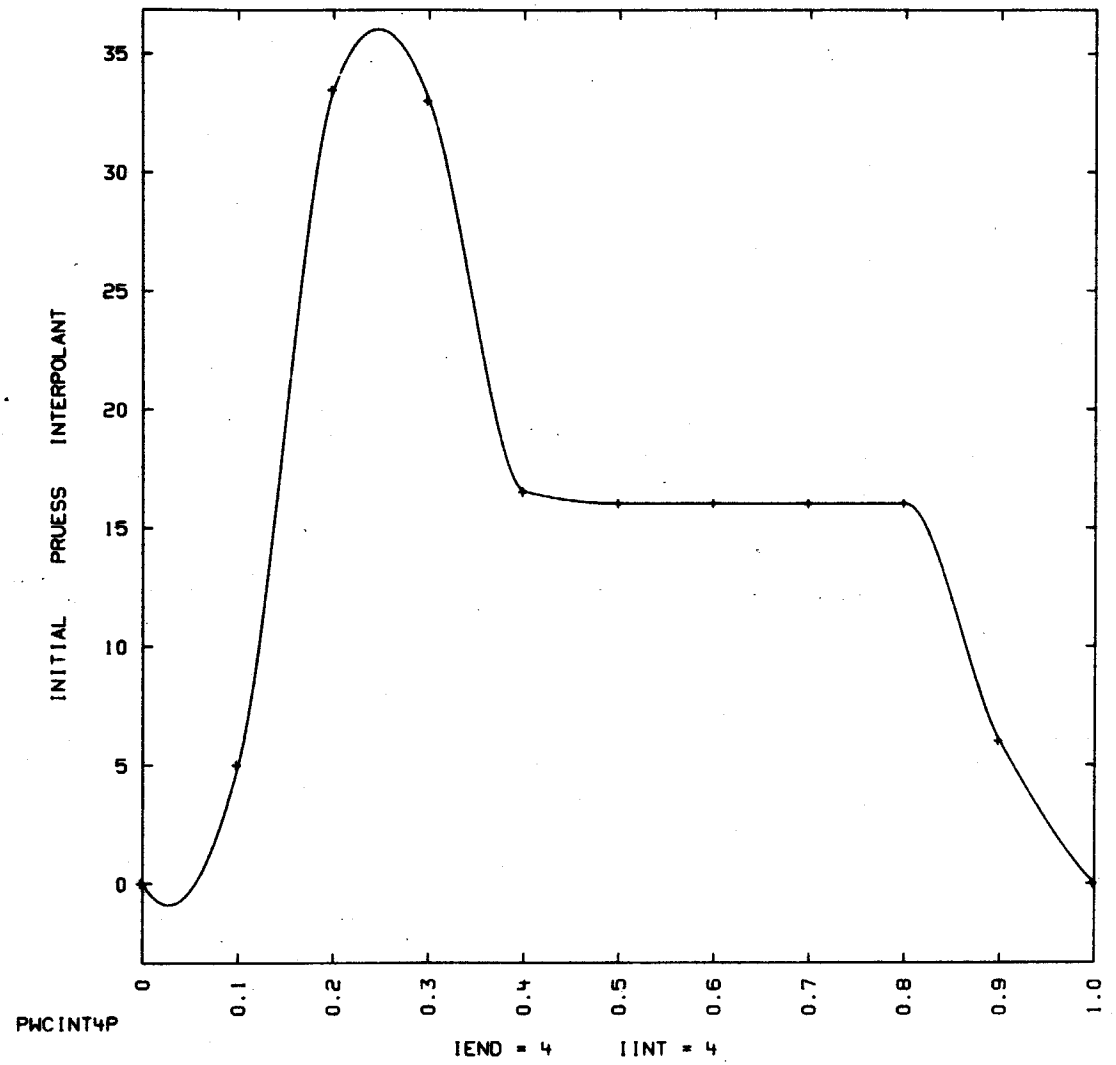



Figure 4.4. Akima Method on Data Set 4.

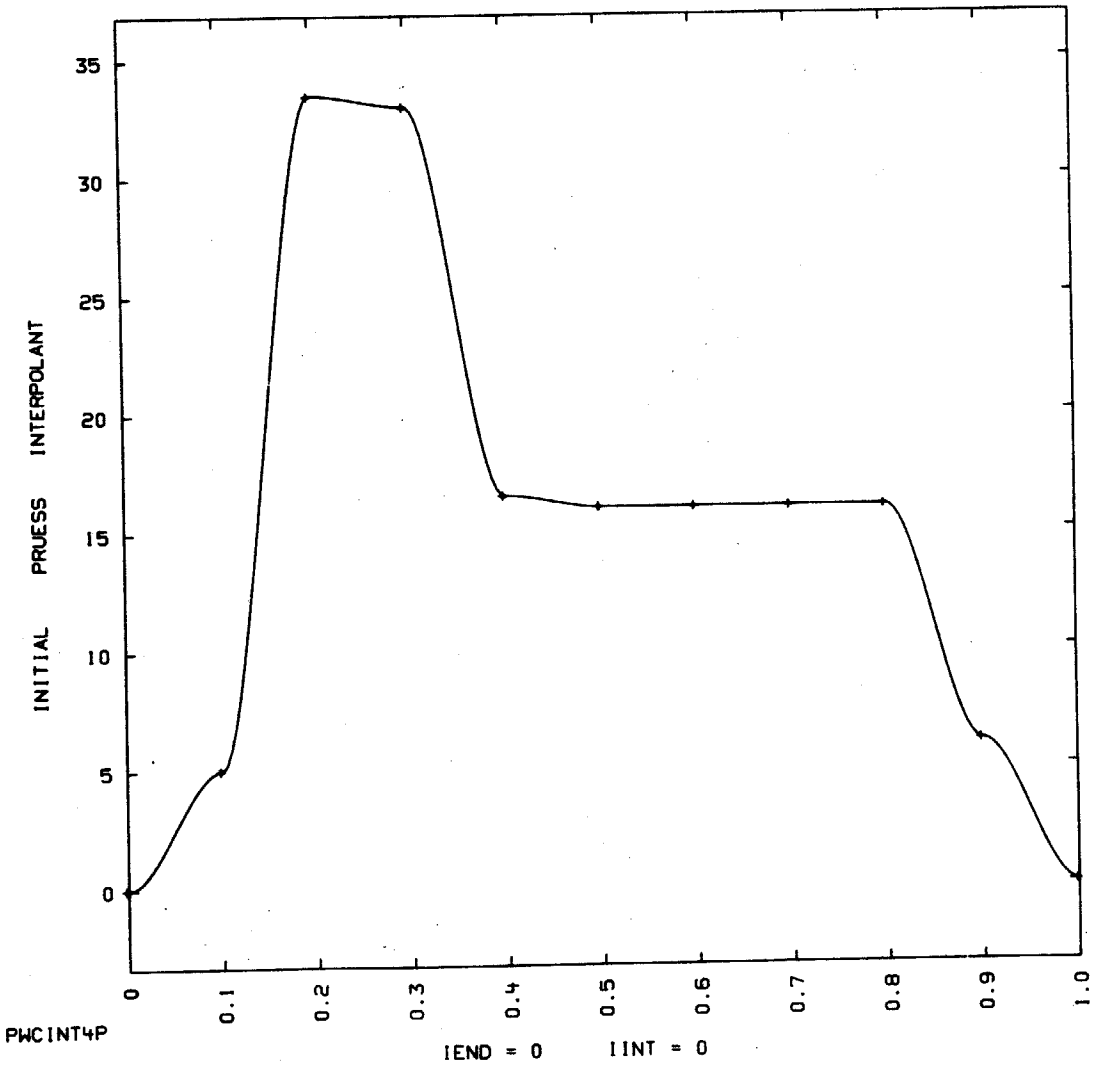



Figure 4.5. Zero Derivatives on Data Set 4.

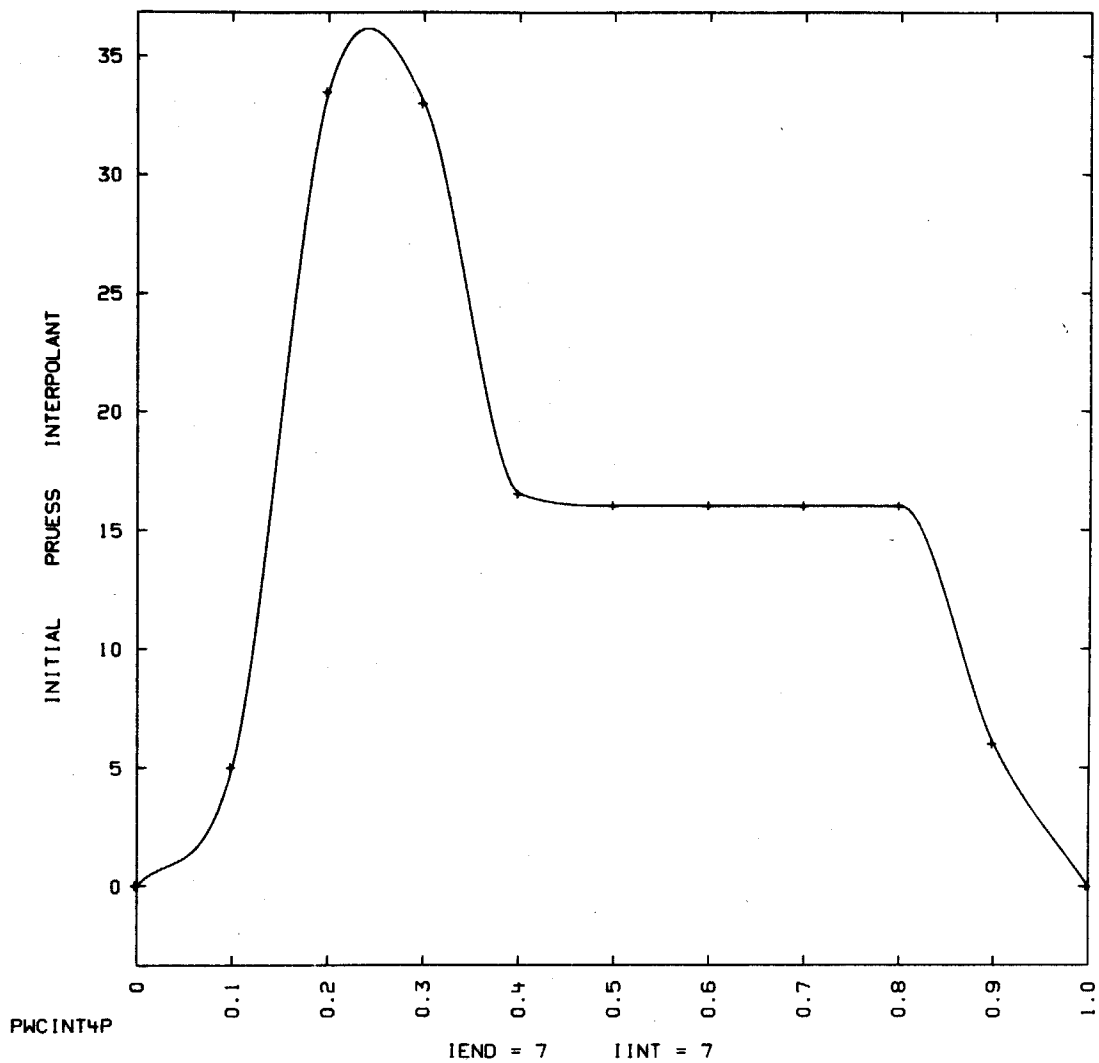



Figure 4.6. Fritsch-Carlson Method on Data Set 4.