Test Problem #4

THIN PLANE LIQUID LAYER PERTURBATION EVOLUTION

Physical formulation

A thin layer is affected by mass force g acting from the top down with boundary pressure values $P=\rho hg$ at the bottom side and P=0 at the upper side. 2D and 3D initial small periodic perturbations of plate shape or velocities are given.

There are analytical solutions in the thin layer approximation (Bakhrakh S.M., Simonov G.P. Proc. 6[th] International Workshop on the Physics of Compressible Turbulent Mixing, 1997, pp. 50-56, pp.57-63). The determinate perturbations exponentially grow, remaining limited, change phase depending on the initial data characterized with dimensionless parameters.

Solutions with given initial perturbations in velocities are most suitable for test computations of perturbation growth. It is convenient to make the computations in Lagrangian variables.

2D problem

Analytical solution. Let x, y- be Cartesian coordinates of the observer's space, t – is time, ξ - is Lagrangian coordinate of the element for which its initial coordinate x_0 – is taken,

$$a = \frac{P}{\rho h_0} = g = \text{const},\ a > 0,$$

where ρ - is density, h_0 - is initial layer thickness, P=const is pressure applied "from below", g - is mass acceleration whose direction is opposite to Y axis.

At the initial time the thin layer occupies position $-h_0/2 < y < h_0/2=0$ and velocity perturbations

$$V_x(0) = r_2 B\cos k\xi\ ,\ V_y(0) = B\sin k\xi$$

are given. The solution for median line is

$$y_0 = \left(\frac{1-r_2}{2}\,\mathrm{sh}(t\sqrt{ka}) + \frac{1+r_2}{2}\sin(t\sqrt{ka}) \right) \frac{B}{\sqrt{ka}} \sin k\xi$$

Here $k = 2\pi/\lambda$ - is the wave number, λ - is wavelength.

Calculation setup. The initial data for the computations are $\rho = 7.8$, $h_0 = 0.1$, $k = 1$, $\lambda = 2\pi$, $B = 0.01$, $a = 1$.

The equation of state is $p = c_0^2(\rho - \rho_0)$, where $c_0 = 4.6$.

Number of cells: 32 for wavelength, 4 for layer thickness.

The computations proceeded till time $t = 5$.

The computations vary $r_2 = -1, 0, 1, 2$.

Output required:

- Peak amplitude vs. time compared to the analytical solution.
  Suggested size of the graph - 7 × 7 cm.
- Relative error in estimated amplitude at $t = 5$

$$\delta = \frac{a_{calc} - a_{theory}}{a_{theory}}.$$

- Shape of one period of the layer median line at $t = 5$.

  Suggested size of the graph - 10 × 10 cm.

3D problem

Analytical solution. Let x, y, z- be Cartesian coordinates of the observer's space, t- is time, ξ, η - are Lagrangian coordinates of the element for which their initial coordinates x_0, y_0 are taken,

$$a = \frac{P}{\rho h_0} = g = \text{const}, \quad a > 0,$$

where ρ - is density, h_0 - is initial layer thickness, P=const is pressure applied "from below", g - is mass acceleration whose direction is opposite to Z axis.

At the initial time the thin layer occupies position $-h_0/2 < z < h_0/2 = 0$ and velocity perturbations

$$V_x(0) = r_2 B \cos k\xi \cos n\eta,$$
$$V_y(0) = -r_2 m B \sin k\xi \sin n\eta,$$
$$Vz(0) = B \sin k\xi \cos n\eta.$$

are given. The solution is

$$z = 0.5[P_1 sh(\omega t) + P_2 \sin(\omega t)]\frac{B}{\omega} \sin k\xi \cos n\eta,$$

Here $k = 2\pi/\lambda_1$, $n = 2\pi/\lambda_2$, , λ_1, λ_2 - are relevant wavelengths.

$$\omega^4 = a^2(k^2+n^2) = a^2k^2(1+m^2), \quad m = n/k,$$

$$P_1 = 1 - r_2(m^2+1)^{0.5}, \quad P_2 = 1 + r_2(m^2+1)^{0.5}.$$

Calculation setup. The initial data for the computations (layer thickness, perturbation amplitude, density) are the same like those for 2D computations, n = k = 1, r_2 = -1, 1, $\frac{1}{\sqrt{2}}$.

The equation of state is $p = c_0^2(\rho - \rho_0)$, where $c_0 = 4.6$.

Number of cells: 4*32*32.

The computations proceeded till time t = 4.

Output required:

- Peak amplitude vs. time compared to analytical solution.
  Suggested size of the graph - 7 × 7 cm.

- Relative error in estimated amplitude at t = 4

$$\delta = \frac{a_{calc} - a_{theory}}{a_{theory}}.$$

- Shape of one period of the layer median surface at t = 4.
  Suggested size of the graph - 10 × 10 cm.

The appendix

The program of random number generation (NUMBER)

```
SUBROUTINE RANDOM(NUMBER)
REAL*4 NUMBER, A
INTEGER IX, IY, IZ
DATA IX/5/, IY/11/, IZ/17/
 IX=171*MOD(IX,177)-2*(IX/177)
 IY=172*MOD(IY,176)-35*(IY/176)
 IZ=170*MOD(IZ,178)-63*(IZ/178)
 IF(IX.LT.0) IX=IX+30269
 IF(IY.LT.0) IY=IY+30307
 IF(IZ.LT.0) IZ=IZ+30323
 A=FLOAT(IX)/30269.+FLOAT(IY)/30307.+FLOAT(IZ)/30323.
 NUMBER=AMOD(A,1.)
RETURN
END
```