An Efficient Orthogonal Finite Element Method for Time Domain Electromagnetics

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Overview

- Vector wave equation
- Time domain finite element method
- Tradition edge elements
- Augmented edge elements
- Preliminary results

Vector wave equation

- Partial differential equation

$$\varepsilon \frac{\partial^2}{\partial t^2} E = -\nabla \times \frac{1}{\mu} \nabla \times E - \frac{\partial}{\partial t} J \;\; in \;\; \Omega \qquad \hat{n} \times E = E_{bc} \;\; on \;\; \Gamma$$

- Variational form

$$\frac{\partial^2}{\partial t^2}(\varepsilon E, T) = \left(\frac{1}{\mu} \nabla \times E, \nabla \times T\right) - \frac{\partial}{\partial t}(J, T)$$

  – solution space $E \in H(curl)$

  – test space $T \in \{v : v \in H(curl); \hat{n} \times v = 0 \;\; on \;\; \Gamma\}$

  – inner product $(u,v) = \int u \bullet v \, d\Omega$

Finite element method

- Linear edge basis functions

$$W_i = N_j \nabla N_k - N_k \nabla N_j$$

- System of ODE's

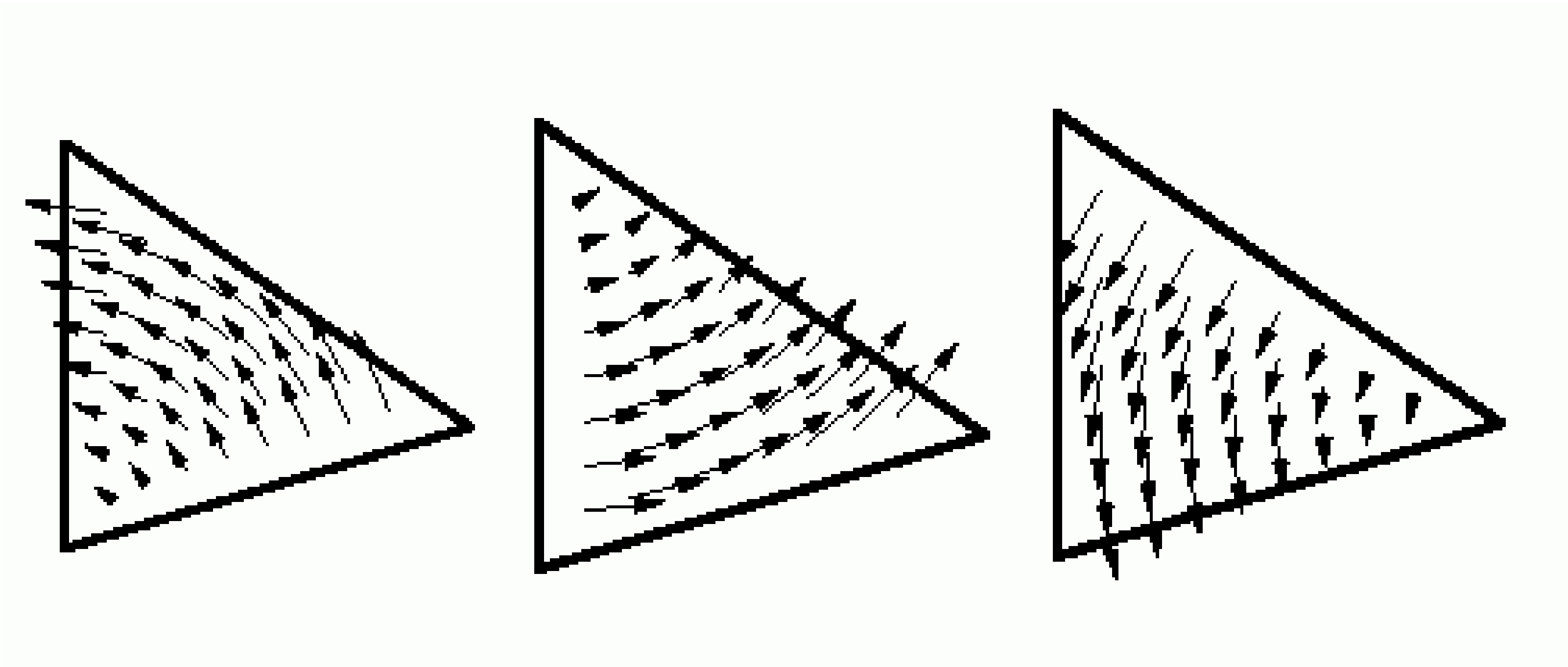
$$A \frac{\partial^2}{\partial t^2} e = Ce + s$$

  - capacitance matrix $A_{ij} = \left(\varepsilon W_i , W_j \right)$
  - inductance matrix $C_{ij} = \left(\frac{1}{\mu} \nabla \times W_i , \nabla \times W_j \right)$
  - *e* is array of degrees of freedom

Edge basis functions are not orthogonal

- Time integration requires the solution of $Ax = b$ at every time step
- Typically, 5-15 ICCG iterations are required to solve the system
- "Mass" lumping or point-matching are options, but results are questionable

Linear edge elements



Construction of augmented basis functions

- Define "bubble" functions

$$B_i = \hat{n}_i N_j N_k$$

  – zero tangential component along mesh edges

  – non-zero normal component for edge *i* only

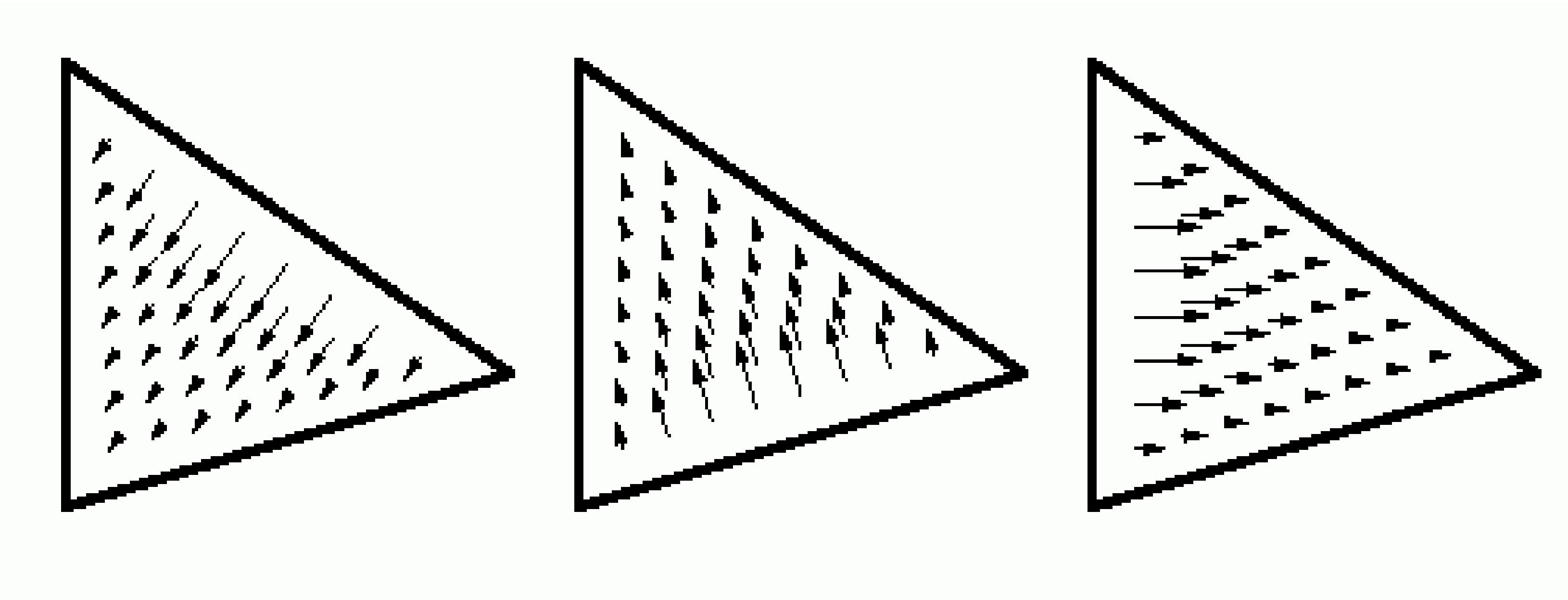
- Define new inner product

$$\langle u,v \rangle = \sum_{i=1}^{3} \alpha_i u(m_i) \bullet v(m_i) \approx (u,v)$$

  – m_i is midpoint of edge *i*

Auxiliary bubble functions

- Bubble functions B_i are local to each cell, total number is 3*(number of cells)



Apply local Gramm-Schmidt

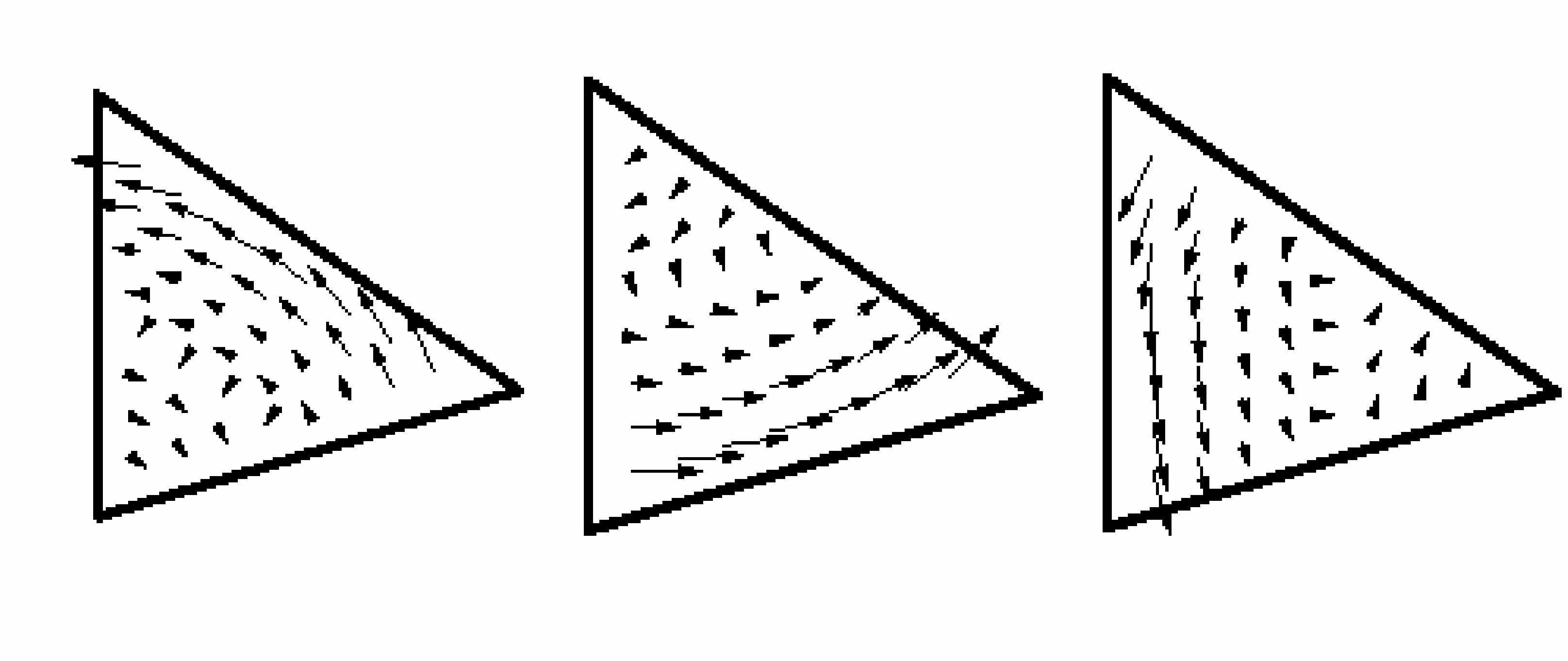
- Define new edge functions

$$Z_i = W_i - \sum_{j=1}^{3} \frac{\langle W_i, B_j \rangle}{\langle B_j, B_j \rangle} B_j$$

  – same tangential continuity as W functions
  – orthogonal to all B functions
  – orthogonal to each other as well

Orthogonal edge elements

- Edge elements Z_i



Approximate variational form

- Use the modified inner product

$$\frac{\partial^2}{\partial t^2}\langle \varepsilon E, T\rangle = \left\langle \frac{1}{\mu}\nabla\times E, \nabla\times T\right\rangle - \frac{\partial}{\partial t}\langle J, T\rangle$$

- The solution space is

$$F_i = Z_i \cap B_i$$

  - the dimension of *F* is 3*(number of cells) + (number of edges)
  - the resulting capacitance matrix is larger, but it is now a diagonal matrix

Advantages

- The capacitance matrix is diagonal
- Since the C matrix representing the curl-curl operator is symmetric, the leapfrog time integration is provably stable
- Since the test space includes the original edge elements, charge is conserved in a variational sense

Computational results

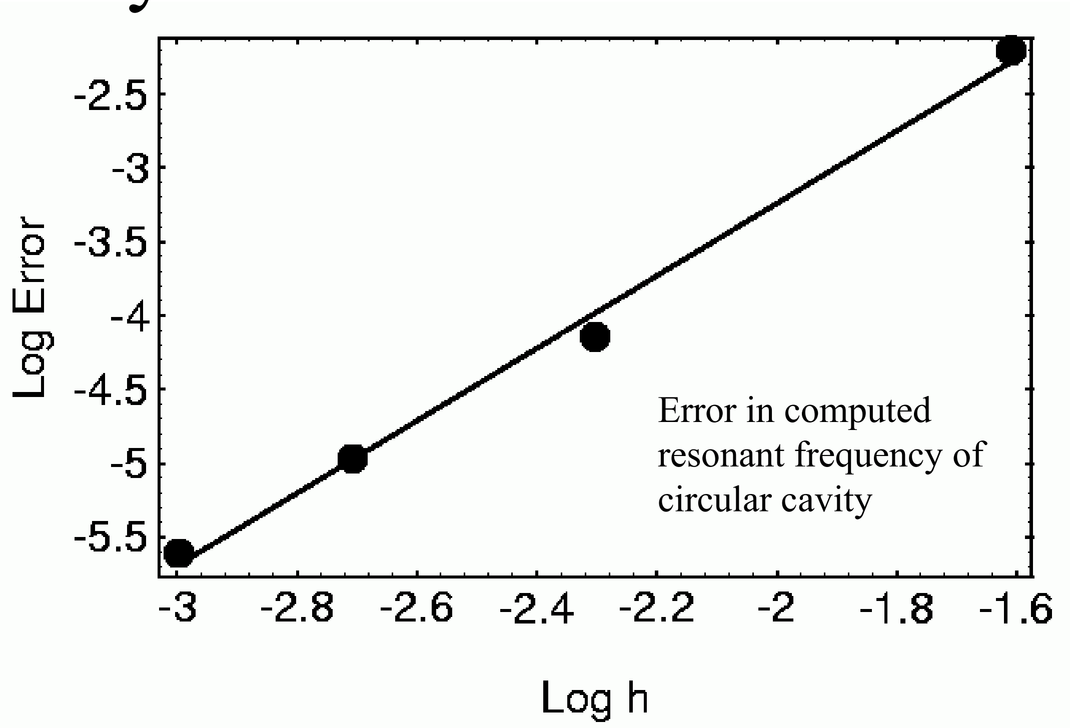
- This approach is approximately 3 times more efficient than the standard approach

		CPU sec original	CPU sec orthogonal
# cells	600	84.9	34.4
	2400	863.5	272.2
	5400	2989	972
	9600	7173	2166

Time to compute resonant frequency of circular cavity

Computational results

- Least-square fit indicates second-order accuracy



Conclusions

- Orthogonal basis functions can significantly reduce CPU time compared to tradition edge basis functions
- Remains second order accurate
- Difficult to generalize to 3D?
- Difficult to generalize to higher order elements?