Generalized High Order Interpolatory 1-Form Bases for Computational Electromagnetics

R. Rieben, G. Rodrigue, D. White*
Lawrence Livermore National Laboratory

1 Introduction

We are concerned with the finite element solution of Maxwell's equations on 3D unstructured grids. While there are a great variety of formulations in use, the Galerkin procedure applied to the curl-curl Helmholtz equation (or the vector wave equation in the time domain) is the most popular. For this equation, the choice of basis functions is of key importance, and there are numerous advantages to Nedelec's curl-conforming (often called called H(curl), edge, or vector) bases. These basis functions allow for a jump discontinuity of electric field across material discontinuities, they are convenient for enforcing the essential PEC boundary conditions and they permit solutions that are free of spurious irrotational modes. These bases were introduced by Nedelec [1], Bossavit [2], Crowley [3] and others, and have successfully been used for multifarious applications including RCS, waveguides, antennas, and optics. Recently Hiptmair [4] has advocated the connection between these curl-conforming bases and differential forms, and this connection is followed here. Curl-conforming basis functions are appropriate for discretization of physical quantities that behave like 1-forms, such as the electric field or the magnetic vector potential.

While the abstract mathematical work such as presented in ([1],[4]) is valid for arbitrary order basis functions, most engineering applications to date have been restricted to first or second order basis functions (following [1] we refer to a basis containing terms of up to order n, as an $n'th$ order basis). This is due to the implicitly defined nature of the basis presented in these mathematical works. Recently, explicit formulae for arbitrary order interpolatory bases were developed [5]. This explicit formulation has spurred the development of many high-order simulation codes. High-order methods can yield extremely accurate and efficient results for certain problems with smoothly curved boundaries. In simple terms, these bases are generated by multiplying the lowest order 1-form basis functions by products of one-dimensional Silvester-Lagrange polynomials. These bases functions are explicit and easily computed, but it is not clear that they are optimal for high-order interpolation.

It is well known that a given function can be approximated to arbitrary accuracy by a polynomial; this is the essence of the Weierstrass Approximation Theorem. However, this theorem *does not* say that arbitrary accuracy can be achieved by a polynomial with uniformly spaced interpolation points. The inability of uniformly spaced interpolatory polynomials to approximate certain functions is often referred to as the Runge phenomenon. While this phenomenon is usually discussed in the context of interpolation of one-dimensional scalar functions, in the next section we demonstrate the same behavior for 1-form basis functions of the type presented in [5]. We also present a solution to this problem, which is to use non-uniformly spaced interpolatory polynomials such as Chebyshev or Gauss-Lobatto polynomials in the construction of 1-form basis functions. It is in this sense that our basis functions are generalized; arbitrary polynomials may be used to construct the 1-form bases, and our procedure reduces to that presented in [5] when Silvester-Lagrange polynomials are used.

In addition, the approach presented here is different from that given in [5] in that the bases are defined on a reference element and then later transformed to an actual distorted element using the Jacobian. This approach yields a more efficient and elegant computer implementation; it does not matter how complex the bases are as they only need to be computed once instead of over and over again for every element. The transformations are based on the properties of differential forms [4]. The distinct rules for the correct transformation of the interpolation vectors, the bases themselves, and the curl of the bases is presented in the next section. In this abstract only hexahedral basis functions are presented.

2 Generalized Construction of 1-form Bases on the Hexahedron

Following [1] an appropriate finite element space for 1-forms on a reference hexahedron is

$$W^{1,p} = \{\vec{u}; u_x \in Q_{p-1,p,p}, u_y \in Q_{p,p-1,p}, u_z \in Q_{p,p,p-1}\} \tag{1}$$

where $Q_{a,b,c}$ denotes the polynomial space in three variables (x, y, z) whose maximum degree is a in x, b in y and c in z. While this is not the starting point of the bases presented in [5], they can be derived from (1) by choosing $Q_{a,b,c}$ to be products of 1D Silvester and Shifted-Silvester polynomials. However other polynomials, perhaps with non-uniform interpolation points, can also be used to construct the space given by (1). One approach for quantifying the efficacy of alternative interpolatory polynomials is to examine the Lebesgue constant of the resulting basis.

In approximation theory, the Lebesgue constant is a well know measure of the "performance"of an interpolatory basis. To be more specific, suppose we have an arbitrary function f and an approximation to this function of order p denoted as $\tilde{f}^p$. The approximating function $\tilde{f}^p$ is constructed by means of an interpolatory basis expansion. We define the absolute error of the function f and its interpolant $\tilde{f}^p$ for some well defined norm as:

$$\Delta f = |f - \tilde{f}^p| \tag{2}$$

Then for stability of the approximation method we require:

$$\lim_{p \to \infty} \Delta f = 0 \tag{3}$$

When the stability requirement of (3) is not met, we observe what is commonly referred to as the Runge phenomenon, i.e the error of our approximation can *increase* as the order of the approximation increases. It is interesting to note that the stability of the approximation is based *solely* on the choice of interpolation points. For an arbitrary interpolating polynomial of order p defined over the set of interpolation points $X = \{x_1, x_2, \ldots, x_{p+1}\}$, we have the following bound:

$$|f - \tilde{f}^p| \leq (1 + \Lambda^p(X))|f - \tilde{f}^p_{best}| \tag{4}$$

where the Lebesgue constant is defined as:

$$\Lambda^p(X) = \max_{x \in [-1,1]} \sum_{i=1}^{p+1} |\mathcal{L}_i^p(x; X)| \tag{5}$$

For Silvester-Lagrange polynomials the Lebesgue constant grows quite rapidly as a function of p, hence the upper bound on the approximation also grows rapidly. Choosing sets of interpolations points X that minimize the Lebesgue constant over the domain $[-1, 1]$ has been the focus of many years of research in approximation theory. For example, Chebyshev points or Gauss-Lobatto points can be used to construct nearly optimal interpolatory polynomials. In Figure 1 we compare the Lebesgue constants for 3D 1-form basis functions defined on a reference hexahedron constructed using the approach from [5] and a new approach using interpolatory polynomials defined by the extended set of zeros of the Chebyshev polynomials. In addition, we compare the approximation error for a specific function (eq. 6). Figure 1 indicates that for large p the use of non-uniformly spaced interpolation points can reduce the worst-case approximation error by several orders of magnitude. Also note that the approximation error for uniformly spaced points diverges to infinity while error for non-uniformly spaced points converges to zero. To our knowledge, this is the first time Lebesgue constants have been computed for 1-form basis functions.

As a specific example to help visualize the approximation error we choose $p = 6$ and compare the Silvester-Lagrange approach from [5] to a new approach using extended Chebyshev polynomials.

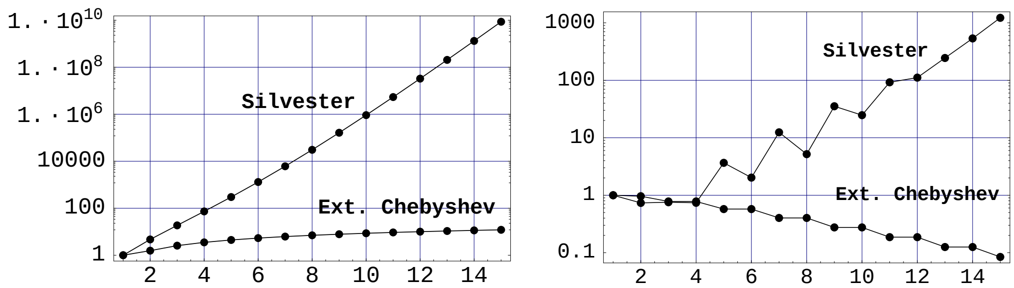

Figure 1: Lebesgue constant vs. p for 3D 1-form basis (*left*) and interpolation error vs. p for eq. 6 (*right*)

In both cases the polynomials are of the Nedelec form given by (1). The vector function we are approximating is

$$\vec{f}(x,y) = \{\frac{1}{(1+x^2)(1+y^2)}, \frac{1}{(1+x^2)(1+y^2)}\} \tag{6}$$

Figure 2 shows the x-component of the function $\vec{f}$, the interpolation $\tilde{f}^p$ using the approach from [5], and the new approach using extended Chebyshev polynomials. Note how the new approach has significantly reduced oscillations at the boundary of the interpolation domain, with a maximum error that is several times smaller than the original approach.

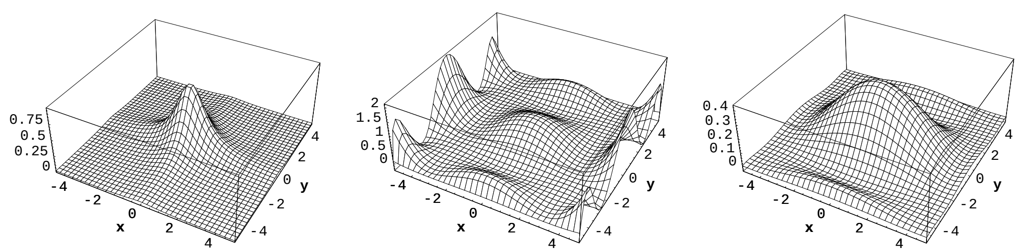

Figure 2: x-component of Exact function (*left*), Shifted Silvester approximation (*middle*) and Extended Chebyshev approximation (*right*)

Now consider a mapping $\vec{y} = \Phi(\vec{x})$ that maps the reference hexahedron to an arbitrary (non-twisted) hexahedron. The order of this mapping need not be the same as the order of the 1-form bases. We define the Jacobian of this mapping as

$$J_{i,j} = \frac{\partial y_j}{\partial x_i} \tag{7}$$

For interpolatory 1-form basis functions the degrees-of-freedom are "interpolation vectors" rather than simple points. These degrees-of-freedom are dual to the basis functions, meaning that $(W_i^{1,p}, b_j) = \delta_{i,j}$ where $W_i^{1,p}$ is the $i'th$ basis function, b_j is the $j'th$ interpolation vector which is zero everywhere except at a single interpolation point, and $(,)$ is an integral in the distributional sense. For 1st order 1-forms on the reference hexahedron the interpolation vectors are tangent to the edges and centered at edge midpoints, for higher order bases these interpolation vectors may lie on the element faces or in the interior of the element. Under the mapping $\vec{y} = \Phi(\vec{x})$ these interpolation vectors transform *contravariantly* and will remain tangent to the edges (or faces) of the element. The 1-form basis functions transform *covariantly* and do not remain tangent to the element edges (or faces). In the finite element solution of Maxwell's equations, the curl of the basis functions are required. The curl of a 1-form is a 2-form, and 2-forms transform according to the Piola transformation [4]. Given these transformations the bases need only be evaluated on the reference element and transformed accordingly. These transformations are summarized in Table 1.

Object	**Transformation**	**Formula**
Interp Vectors	Contravariant	$\hat{b} = J^T b$
Basis Functions	Covariant	$\hat{W} = J^{-1} W$
Curl of Basis	Piola	$d\hat{W} = \frac{1}{\lvert J \rvert} J^T dW$

Table 1: Summary of transformations

3 Conclusions

The explicit formulae for arbitrary order 1-form interpolatory bases developed in [5] are based upon Silvester-Lagrange polynomials. It is well known that these polynomials have the potential to exhibit erratic behavior as the order p increases due to their rapidly increasing Lebesgue constants. In this paper we present computed Lebesgue constants for these 1-form bases and for new bases that utilize interpolation points based on the zeros of Chebyshev polynomials. These new bases have significantly smaller Lebesgue constants (near optimal), and we show by example that this directly affects the interpolation error. We show that for large p, the interpolation error for non-uniform bases can be orders of magnitude smaller than that of uniform bases. The procedure presented here is generic in the sense that *any* interpolatory polynomial can be used. This generality is achieved by constructing the 1-form basis function on a reference element and transforming to the actual element using appropriate transformation rules. The appropriate transformation rules are conceived by identifying the interpolation vectors as tangent vectors, the basis functions as 1-forms and the curl of the basis functions as 2-forms.

References

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